

Mortality Risk, Insurance, and the Value of Life*

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Abstract. Public health insurance and public annuity programs account for half of federal spending, and have traditionally been viewed as unconnected. We develop an economic framework for valuing improvements in health and apply it to exploring the relationship between annuity programs and the value of life. Incorporating incomplete annuitization into the conventional economic theory of life-extension generates several novel findings. First, public annuity programs have boosted the demand for life-extension. For instance, US Social Security added \$11.5 trillion (10.6 percent) to the value of post-1940 longevity gains, which adds nearly 50 percent to the aggregate insurance value provided by Social Security's pensions. Second, in contrast to the conventional theory, a given mortality improvement may be worth more, not less, to patients facing shorter lives. Holding income and wealth constant, the value of statistical life (VSL) for a 50-year-old with 15 years of remaining life expectancy is \$1 million greater than it is for a 50-year-old with 35 years of life expectancy. Thus, existing economic analysis may be undervaluing treatment of severe illness relative to mild ones. This result also reconciles an empirical puzzle with the economic approach to valuing life, because consumers often report a preference for extending life among those with the bleakest survival prospects. Finally, our framework implies that treatments may be worth more than prevention, all else equal. For instance, we calculate that treating a chronic condition such as cancer is worth 50 to 100 percent more than achieving the same outcome through prevention.

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I. INTRODUCTION

The economic analysis of risks to life and health has made enormous contributions to both academic discussions and public policy. Economists have used the standard tools of life-cycle consumption theory to propose a transparent framework that measures the value of improvements to both health and longevity (Arthur 1981; Rosen 1988; Murphy and Topel 2006). Economic concepts such as the value of statistical life now play central roles in public policy discussions surrounding investments in medical care, public safety, workplace safety, environmental hazards, and countless other arenas.

The standard framework has typically assumed complete annuitization and deterministic mortality risk. While analytically convenient and useful for illustrating some of the underlying economics, these assumptions hamper the model's predictive power in several ways when studying individual behavior and the relationships between alternative mechanisms for risk-reduction. In addition, they gloss over policy-relevant relationships between the value of life and the structure of the annuity market, and cannot meaningfully distinguish between preventive care and therapeutic care.

Complete annuity markets shield an individual's consumption against mortality risk. An incompletely annuitized consumer will have a different consumption profile, which in turn affects her value of life. A very simple example illustrates the intuition. Imagine a 60-year-old retiree with no bequest motive and a flat optimal consumption profile. If she fully annuitizes her savings, her consumption remains flat at, say, \$30,000 annually. Now suppose she cannot annuitize any of her wealth. It is well known that in this case it is optimal to shift consumption forward (Yaari 1965), because consumption allocated to later time periods will not be enjoyed in the event of an early death (see Figure 1). An important insight of our paper is that—at least for some initial period of time—mortality risk increases consumption, which reduces the *marginal utility* of consumption. We demonstrate that although an increase in mortality risk always reduces lifetime utility, the accompanying reduction in the contemporaneous marginal utility of consumption can be large enough to cause the value of statistical life to *increase*. This is in stark contrast to the conventional model with full annuitization, where an increase in mortality always reduces the value of statistical life.

For the same underlying reason, an adverse stochastic *shock* to mortality can also cause VSL to rise. Our model demonstrates that the effect of a mortality shock on the value of life depends on two preference parameters: the elasticity of intertemporal substitution, which measures the curvature of the utility function, and prudence, which measures the curvature of the marginal utility function. An adverse mortality shock increases the value of life when demand for current consumption is sufficiently inelastic, or when the marginal utility of demand is sufficiently linear. Moreover, the value of life varies with the size of mortality shocks when consumers are incompletely annuitized. This contrasts with the standard implication that a given reduction in mortality will be equally valuable, regardless of a consumer's baseline mortality risk. Instead, we show that the value of statistical life is frequently higher for an individual diagnosed with a more fatal illness, and vice-versa. For example, we calculate using real-world data that, holding income and wealth constant, VSL varies from \$3.5 million to \$4.5 million for 50-year-olds with 35 to 15 years of remaining life expectancy, respectively. Under incomplete annuitization, it can often be more valuable to save shorter lives than longer ones. This insight is consistent with data on how consumers view the value of life-extension (Nord et al. 1995; Green and Gerard 2009; Linley and Hughes 2013), and can better inform the way health economists and healthcare payers assess the value of medical technologies.

Our analysis illustrates the connections between public annuity programs and the societal value of mortality reductions. For example, we calculate that US Social Security added \$11.5 trillion (10.6 percent) to the value of post-1940 longevity gains, which adds nearly 50 percent to the aggregate insurance value provided by Social Security's pensions. Social Security has also increased the aggregate

value of reducing future mortality risks by over 10 percent, so that a 1 percent reduction in population-wide mortality is \$143 billion more valuable than it would have been without a Social Security pension. And, increase the size of Social Security pensions by 50 percent would add a further \$49 billion of value to this mortality decline. Intuitively, annuitization tends to raise the value of life-extension at older ages where people might otherwise have outlived their wealth. Since the absolute number of deaths is quite high at those ages, this also tends to boost the value of proportional reductions in mortality.

Finally, our framework takes the more realistic perspective that an individual faces uncertainty over her future mortality risk. Allowing mortality to be stochastic produces additional insights. The conventional model quantifies the value of statistical life, but it has little to say about the continuum of health events that precede death. Our framework lends itself naturally to a more general concept, the value of statistical illness (VSI), which quantifies an individual's willingness to pay to avoid an increase in the risk of acquiring an illness that affects her mortality rate. This allows for the first time an economic comparison of the value of prevention to the value of treatment. In contrast to the conventional model, we show how the value of treatment technologies, which are used after an illness occurs, may differ from the value of preventive technologies, even when both increase life expectancy by identical amounts. This is because treatments are administered in bleaker health states than preventive investments. Employing real-world data from a well-established microsimulation model of health, we calculate that the value of treating life-threatening conditions like cancer is worth 50 to 100 percent more than equivalent preventive treatments that add the same number of years to an individual's life expectancy.

Our study connects the vast literature on the value of life (Arthur 1981; Rosen 1988; Murphy and Topel 2006; Hall and Jones 2007) with the literature on annuities and life-cycle consumption models that goes back to Yaari (1965). It is well known that annuitization provides substantial value by insulating individuals from consumption risk. We show that it also increases the value of statistical life at older ages, and the value of mortality reductions in the aggregate. Our results suggest that more attention should be paid to the public finance interactions between pension and healthcare systems.

This result also has important implications for the economic analysis of medical technology, which explicitly governs the allocation of healthcare resources in many "single-payer" countries, including the United Kingdom, Canada, and Australia (Dranitsaris and Papadopoulos 2015). And, the importance of economic value assessment in the multi-payer US healthcare marketplace continues to grow (Goldman, Nussbaum, and Linthicum 2016). The standard theory asserts that the value of extending life is insensitive to the severity of illness. For instance, the current theory implies equivalence between providing X aggregate life-years to a very large population of hypertension patients and providing major breakthroughs that add X aggregate life-years by extending life substantially for a proportionally smaller population of cancer patients. This equivalence appears to be incorrect without complete annuitization. Thus, our existing approach to healthcare resource allocation underinvests in the treatment of the most life-threatening illnesses relative to less severe conditions.

To our knowledge, our study is the first to provide an economic analysis of both prevention and treatment using a standard life-cycle model. Cost-effectiveness analysis, which is widely employed by healthcare systems across the world to govern the use of new medical technologies, traditionally values prevention and treatment equally (Drummond et al. 2005a). However, we demonstrate theoretically that prevention and treatment are not equally valued. In contrast to the old adage, we calculate that treatment is often significantly more valuable than prevention.

Extending the value of life analysis to a stochastic mortality setting requires us to rely on tools from continuous-time stochastic optimal control. To derive the expressions for VSL and VSI, we rely on a "stochastic" version of the Pontryagin maximum principle following recent ideas from the systems and

control literature (Parpas and Webster 2013). The resulting expressions for VSL generalize the deterministic versions in the earlier literature (Shepard and Zeckhauser 1984; Murphy and Topel 2006), and VSI can in turn be interpreted as a generalization of the concept of VSL. We view our application of these tools as a useful demonstration for other researchers working in stochastic settings.

Section II reviews the predictions of the conventional model for the returns to life-extension and demonstrates how relaxing the perfect annuity assumption alters the predictions of a model with deterministic mortality. Section III generalizes the framework further by incorporating the more realistic assumption of stochastic mortality. Section IV presents empirical analysis that: (1) estimates the effect of public annuity programs on the value of mortality risk-reduction; (2) quantifies how health shocks change the value of statistical life when annuity markets are incomplete; (3) illustrates how more severe health shocks cause consumers to place higher value on a given mortality reduction; and (4) calculates the value of preventing different kinds of illness. Section V concludes.

II. THE VALUE OF LIFE WHEN MORTALITY IS DETERMINISTIC

Consider an individual who faces a mortality risk. We are interested in analyzing the value of a marginal reduction in this risk. We first quantify this value in the conventional setting where markets are complete and the consumer has access to actuarially fair annuities (Rosen 1988; Murphy and Topel 2006). We then repeat this exercise in a “Robinson Crusoe” economy where the consumer cannot purchase annuities to insure against her uncertain lifetime (Shepard and Zeckhauser 1984; Ehrlich 2000; Johansson 2002). We compare our findings for these two polar cases to illustrate the basic insights of the paper. We focus on improvements in longevity and their relationship to annuity insurance markets, but allow for improvements in quality of life as well. Section III then extends the model to accommodate stochastic mortality and introduces the value of statistical illness.

Although it is optimal for a consumer to fully annuitize, real-world annuitization rates are quite low. This “annuity puzzle” is the subject of numerous papers. Many explanations have been suggested, but there is no consensus on what drives incomplete annuitization (Brown et al. 2008). Our model takes the low rate of annuitization as a given empirical fact and illustrates its significance for the value of life. Section IV uses a numerical model to probe the sensitivity of our results to different assumptions about consumer preferences, such as the presence of a bequest motive, which prior studies have argued might rationalize low observed rates of annuitization. There continues to be debate over why real-world consumption profiles and annuity purchase decisions look the way they do. However, as we show, the implications for life-extension depend primarily on the real-world consumption profiles themselves, not the reasons that lie beneath.

II.A. The fully annuitized value of life

Let $c(t)$ be consumption at time t , W_0 be baseline wealth, $m(t)$ be exogenously determined income, ρ be the rate of time preference, and r be the rate of interest.¹ Finally, define $q(t)$ as health-related quality of life at time t . Since it sacrifices little generality in our application, we take the life-cycle quality of life profile $q(t)$ as exogenous. As needed, one can consider any relevant quality of life profile in concert with a given profile of mortality. The maximum lifespan of a consumer is T , and her mortality (hazard) rate at

¹ It is straightforward to incorporate endogenous labor supply (Murphy and Topel 2006). One could also allow income to depend on quality of life or mortality risk. This would have no qualitative effect on the results we present for the fully annuitized model. In the uninsured model presented later in the paper, it would reduce the value of life for working-age individuals who fall ill.

any point in time is given by $\mu(t)$, where $0 \leq t \leq T$. The probability that a consumer will be alive at time t is:

$$S(t) = \exp \left[- \int_0^t \mu(s) ds \right]$$

At time $t = 0$, the consumer fully annuitizes. We assume that annuitization is actuarially fair. The consumer's maximization problem is:

$$\begin{aligned} V(0) &= \max_{c(t)} \int_0^T e^{-\rho t} S(t) u(c(t), q(t)) dt \\ \text{s. t. } &\int_0^T e^{-rt} S(t) c(t) dt = W_0 + \int_0^T e^{-rt} S(t) m(t) dt \end{aligned}$$

The consumer's utility function, $u(c(t), q(t))$, depends on both consumption and health-related quality of life. We assume $u(\cdot)$ is strictly increasing and concave in its first argument, and twice continuously differentiable. Let $u_c(\cdot)$ denote the marginal utility of consumption. Associating the multiplier θ with the wealth constraint, optimal consumption is characterized by the first-order condition:

$$\frac{\partial V}{\partial W} = \theta = e^{(r-\rho)t} u_c(c(t), q(t))$$

To analyze the value of life, let $\delta(t)$ be a perturbation on the mortality rate with $\int_0^T \delta(t) dt = 1$, and consider

$$S^\varepsilon(t) = \exp \left[- \int_0^t (\mu(s) - \varepsilon \delta(s)) ds \right], \varepsilon > 0$$

Let $c^\varepsilon(t)$ represent the equilibrium variation in $c(t)$ caused by this perturbation. As shown in Rosen (1988), the marginal utility of this life-extension is given by

$$\begin{aligned} \left. \frac{\partial V}{\partial \varepsilon} \right|_{\varepsilon=0} &= \left. \frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} S^\varepsilon(t) u(c^\varepsilon(t), q(t)) dt \right|_{\varepsilon=0} \\ &= \int_0^T [e^{-\rho t} u(c(t), q(t)) + e^{-rt} \theta (m(t) - c(t))] \left[\int_0^t \delta(s) ds \right] S(t) dt \end{aligned}$$

The marginal value of life-extension is equal to the marginal rate of substitution between longer life and wealth:

$$\frac{\partial V / \partial \varepsilon}{\partial V / \partial W} = \int_0^T e^{-rt} S(t) \left(\frac{u(c(t), q(t))}{u_c(c(t), q(t))} + m(t) - c(t) \right) \left[\int_0^t \delta(s) ds \right] dt \quad (1)$$

The value of a life-year is the value of a one-period change in survival from the perspective of current time:

$$v(t) = \frac{u(c(t), q(t))}{u_c(c(t), q(t))} + m(t) - c(t) \quad (2)$$

The value of a life-year, $v(t)$, is equal to the value of consumption in that year plus net savings, $m(t) - c(t)$. The net savings term is a consequence of the requirement that annuities be actuarially fair. The value of a life-year can be rewritten as:

$$v(t) = m(t) + c(t) \left(\frac{u(c(t), q(t))}{c(t)u_c(c(t), q(t))} - 1 \right) = m(t) + c(t)\phi(c, q)$$

where $\phi(c, q)$ represents the consumer surplus value per unit of consumption. It is positive if average utility exceeds marginal utility. A life-year adds value through two different channels: an increase in earnings, which can finance additional consumption, and an increase in consumer surplus.²

A canonical choice for $\delta(\cdot)$ in equation (1) is the Dirac delta function, so that the mortality rate is perturbed at $t = 0$ and remains unaffected otherwise. This then yields an expression that is commonly called the value of statistical life (VSL):

$$VSL \equiv \int_0^T e^{-rt} S(t)v(t)dt \quad (3)$$

VSL corresponds to the value that the individual places on a marginal reduction in risk of death in the current period. For example, it is the amount that 1,000 people would be collectively willing to pay to eliminate a current risk that is expected to kill one of them. It is equal to the present discounted value of lifetime consumption, plus the change in net savings. Holding wealth constant, VSL increases with survival, which implies increasing returns in health improvements (Murphy and Topel 2006). Conversely, this leads to the conventional wisdom that VSL falls when mortality rises.

The value of statistical life depends on how substitutable consumption is at different ages, i.e., on how easily an individual can reallocate consumption over time. Define the elasticity of intertemporal substitution as:

$$\frac{1}{\sigma} \equiv - \frac{u_{cc}c}{u_c}$$

In addition, define the elasticity of quality of life with respect to the marginal utility of consumption as:

$$\eta \equiv \frac{u_{cq}q}{u_c}$$

When this term is positive, the marginal utility of consumption is higher in healthier states, and vice-versa. Taking logarithms of the first-order condition for consumption and differentiating with respect to time yields the rate of change for consumption over the life cycle:

$$\frac{\dot{c}}{c} = \sigma(r - \rho) + \sigma\eta \frac{\dot{q}}{q} \quad (4)$$

If one assumes that $r > \rho$, and that the marginal utility of consumption is higher when health status is better, then life-cycle consumption will have the inverted U-shape observed in real-world data.³

² Positive consumer surplus may require that consumption remain above a “subsistence” level, $\underline{c} > 0$.

Note the crucial feature of the conventional model that consumption growth over the life-cycle is independent of mortality risk, because the individual is fully insured against that risk. This feature in turn implies that the rate of change in the value of a life-year is also not a function of mortality risk:

$$\frac{\dot{v}}{v} = \left(\frac{1}{\sigma v} \frac{u}{u_c} \right) \frac{\dot{c}}{c} + \left(\frac{-\eta}{v} \frac{u}{u_c} + \frac{q}{v} \frac{u_q}{u_c} \right) \frac{\dot{q}}{q} + \frac{\dot{m}}{v}$$

In sum, we have identified two major features of the conventional, fully annuitized and deterministic model of mortality:

- The relative value of a life-year within a lifetime is independent of mortality risk.
- The value of statistical life falls when mortality rises.

II.B. The uninsured value of life

To illustrate the effects of annuitization, we consider a model without any annuitization possibilities. In our numerical exercises later, we will consider various partial annuitization schemes. To characterize the model without annuitization, we employ the Yaari (1965) model of consumption behavior under mortality risk. The consumer's maximization problem is:

$$\begin{aligned} V(0, W(0)) &= \max_{c(t)} \int_0^T e^{-\rho t} S(t) u(c(t), q(t)) dt \\ \text{s. t. } W(0) &= W_0, \\ W(t) &\geq 0, W(T) = 0 \\ \dot{W} &= rW(t) + m(t) - c(t) \end{aligned}$$

If the non-negative wealth constraint binds, then the solution to the consumer's problem is to set $c(t) = m(t)$. Otherwise, the solution is to maximize subject to the constraint on the law of motion for wealth. We focus here on the latter, nontrivial case.

Optimal consumption is again characterized by the first-order condition:

$$\frac{\partial V(t, W(t))}{\partial W(t)} = \theta = e^{(r-\rho)t} S(t) u_c(c(t), q(t))$$

Unlike in the case of perfect markets, the survival function enters the consumer's first-order condition for optimal consumption. Instead of setting the discounted marginal utility of consumption equal to the marginal utility of wealth, the consumer sets the *expected* discounted marginal utility of consumption at time t equal to the marginal utility of wealth. This effectively shifts consumption to earlier ages in the life-cycle. This is rational because consumption allocated to later time periods will not be enjoyed in the event of an early death.

The expression for the marginal utility of life-extension is:

$$\frac{\partial V}{\partial \varepsilon} \Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} S^\varepsilon(t) u(c^\varepsilon(t), q(t)) dt \Big|_{\varepsilon=0}$$

³ Consumption climbs early in life as the benefits to savings diminish. It declines later in life when quality of life deteriorates. This inverted U-shape for the age profile of consumption has been widely documented across different countries and goods (Carroll and Summers 1991; Banks et al. 1998; Fernandez-Villaverde and Krueger 2007).

$$\begin{aligned}
&= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds \right] S(t) u(c(t), q(t)) dt + \int_0^T e^{-\rho t} S(t) u_c(c(t), q(t)) \frac{\partial c^\varepsilon(t)}{\partial \varepsilon} \Big|_{\varepsilon=0} dt \\
&= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds \right] S(t) u(c(t), q(t)) dt + \theta \frac{\partial}{\partial \varepsilon} \int_0^T e^{-rt} c^\varepsilon(t) dt \\
&= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds \right] S(t) u(c(t), q(t)) dt,
\end{aligned}$$

where the last equality follows from application of the budget constraint.⁴

Dividing this result by the marginal utility of wealth, θ , then yields the marginal value of life-extension:

$$\begin{aligned}
\frac{\partial V / \partial \varepsilon}{\partial V / \partial W} &= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds \right] S(t) \frac{u(c(t), q(t))}{u_c(c(0), q(0))} dt \\
&= \int_0^T e^{-rt} \left[\int_0^t \delta(s) ds \right] \frac{u(c(t), q(t))}{u_c(c(t), q(t))} dt
\end{aligned} \tag{5}$$

In this setting, the value of a life-year from the perspective of current time is:

$$v(t) = \frac{u(c(t), q(t))}{u_c(c(t), q(t))} \tag{6}$$

When the consumer is uninsured, the value of a life-year depends only on the value of consumption. The net savings term is absent in equation (6) because life-extension has no effect on the consumer's budget constraint.⁵

Choosing again the Dirac delta function for $\delta(\cdot)$ yields an expression for VSL that differs from the perfect markets case:

$$VSL = \int_0^T e^{-rt} v(t) dt \tag{7}$$

The value of statistical life is proportional to (expected) lifetime utility, and inversely proportional to the marginal utility of consumption. It is well known that removing annuity markets lowers lifetime utility (Yaari 1965). As we show more formally below, removing these markets also shifts consumption to earlier ages, thereby lowering the marginal utility of consumption, at least at those ages. When consumers shift consumption forward, the near-term life-years rise in value but distant life-years fall in value. Thus, the net effect of annuity markets on VSL is in general ambiguous. Put differently, exposure to longevity risk does not necessarily lower VSL. In the next section, we will show that this basic insight extends to exposing a consumer to a mortality "shock." We emphasize that in both cases the result depends critically on whether consumers are fully annuitized.

⁴ The budget constraint $W(T) = 0$ implies $\int_0^T e^{-rt} c^\varepsilon(t) dt = W_0 + \int_0^T e^{-rt} m(t) dt$, a value which does not depend on survival and thus is unaffected by life extension.

⁵ Unless the consumer survives until period T , she will die with positive wealth. Although this remaining wealth has no value to an individual with no bequest motive, it may be of value to society. When calculating the *social* value of life-extension, we account for the effect of increased longevity on bequests by including a net savings term, defined to be the expected increase in future earnings net of consumption, as in equation (2). This term reflects the external effect on society's aggregate wealth due to increased longevity.

Unlike the perfect markets case, the life-cycle consumption profile of the non-annuitized individual depends explicitly on mortality risk. Taking logarithms of the first-order condition for consumption and differentiating with respect to time yields:

$$\frac{\dot{c}}{c} = \sigma(r - \rho) + \sigma\eta \frac{\dot{q}}{q} - \sigma\mu(t) \quad (8)$$

Comparing this result to the standard case, given by equation (4), reveals both similarities and differences. As in the standard, fully annuitized model, the non-annuitized consumption profile described by equation (8) changes shape when the rate of time preference is above or below the rate of interest and when the quality of life changes. Unlike in the standard model, the consumption profile described by equation (8) depends explicitly on the mortality rate, $\mu(t)$. Higher rates of mortality depress the rate of consumption growth over the life-cycle. This rate of growth is always higher in the fully annuitized case, in which the last term drops out of the consumption growth equation (8). Put another way, removing the annuity market “pulls consumption earlier” in the life-cycle.

An appealing feature of the uninsured model is that it generates an inverted U-shape for the profile of consumption under quite natural assumptions. Low income early in life and high mortality risk later in life are sufficient conditions for the inverted U-shape consumption profile. One need not impose the ad hoc assumptions on the signs of $r - \rho$ or η that are necessary in the fully annuitized model (Murphy and Topel 2006).

The life-cycle profile of the value of a life-year is:

$$\frac{\dot{v}}{v} = \left(\frac{1}{\sigma} + \frac{c}{v}\right) \frac{\dot{c}}{c} + \left(\frac{qu_q}{u} - \eta\right) \frac{\dot{q}}{q} \quad (9)$$

An important implication of (9) is that willingness to pay for longevity depends on the life-cycle mortality profile because of its dependence on the rate of change in consumption. Holding quality of life constant, it is evident from equation (6) that increases in the mortality rate—which shift consumption forward—will raise v , the current value of a life-year. That is, mortality also shifts forward the value of life. All else equal, individuals who face high mortality risks will pay more for a marginal (near-term) life-year, but less for a distant life-year, than healthy peers who face low mortality risks. This differs from the implications of the conventional model, in which higher mortality reduces the values of life-years but has no impact on their relative values.

At the aggregate level, as societies become richer and live longer, the fraction of wealth spent on health will depend not just on the income elasticity of health, but also on the degree of survival uncertainty they face. Furthermore, our results imply that public programs that increase annuitization rates, such as Social Security, will affect society’s willingness to pay for longevity, thereby creating a feedback loop that could dampen or increase program expenditures.⁶ In our numerical exercises, we will quantify how the degree of annuitization influences the value of statistical life.

To summarize the findings for this uninsured model, we have identified the following two properties that contrast with those of the fully annuitized model:

- The values of near-term life-years rise, and distant life-years fall, when mortality rises;
- The value of statistical life may rise or fall when mortality rises.

⁶ Philipson and Becker (1998) make the important, but distinct, point that the moral hazard effects of public annuity programs also increase an individual’s willingness to pay for longevity gains.

In the next section, we allow mortality to be stochastic so that we can investigate formally the effect of disease and other health shocks on the value of life. Before turning to that analysis, we pause to note that suffering a health shock is similar to removing access to annuity markets, which exposes an individual to mortality risk. We have shown here that this shifts the value of life-years forward, with an ambiguous net effect on VSL. As we shall see, health shocks have a similar effect.

III. THE VALUE OF LIFE WHEN MORTALITY IS STOCHASTIC

The previous analysis demonstrates that mortality risk affects the value of life when annuity markets are incomplete. Prior studies have overlooked this relationship by assuming complete annuitization. However, the conventional framework is ill-equipped to study the influence of mortality risk for another reason as well. Prior analysis, just like our deterministic model above, treats the mortality rate as a nonrandom parameter (cf. Murphy and Topel, 2006). Thus, shifts in mortality risk reflect preordained and anticipated changes in mortality. In the real world, however, neither the timing nor the size of shifts in mortality risk is known. As a related matter, the conventional framework does not allow for different health states. This omission precludes a meaningful analysis of the value of preventing health deterioration.

This section extends our analysis to allow for stochastic mortality. Specifically, we assume that the mortality rate now depends on the individual's health state. Let Y_t be a continuous-time Markov chain with finite state space $Y = \{1, 2, \dots, n\}$. Denote the transition intensities by:

$$\lambda_{ij}(t) = \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{P}[Y_{t+h} = j | Y_t = i], j \neq i,$$

$$\lambda_{ii}(t) = - \sum_{j \neq i} \lambda_{ij}(t)$$

The mortality rate at time t is defined as

$$\mu(t) = \sum_{j=1}^n \bar{\mu}_j(t) \mathbf{1}\{Y_t = j\}$$

where $\{\bar{\mu}_j(t)\}$ are exogenous and $\mathbf{1}\{Y_t = j\}$ is an indicator variable equal to 1 if the individual is in state j at time t and 0 otherwise. Without meaningful loss of generality, we assume that individuals can transition only to higher-numbered states, i.e., $\lambda_{ij}(t) = 0 \forall j < i$, so that the probability that a consumer in state i at time 0 remains in state i at time t is equal to:⁷

$$\tilde{S}(i, t) = \exp \left[- \int_0^t \left(\bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) \right) ds \right]$$

⁷ That is, an individual can transition from state i to j , $i < j$, but not vice versa. This does not meaningfully limit the generality of our model, because one can always define a new state $k > j$ where $\bar{\mu}_k(t) = \bar{\mu}_i(t) \forall t$.

A complete annuities market allows the consumer to insure fully against mortality risk even when mortality is stochastic.⁸ Appendix C provides a full derivation for a setting with complete markets and demonstrates that stochastic mortality, by itself, does not alter the basic insights regarding VSL offered by the prior literature as long as one maintains the assumption of full annuitization. Appendix C also derives expressions for the value of preventing illness when the consumer is fully annuitized. We defer discussion of those results until later in this section.

Here, we focus on the uninsured case. The consumer's maximization problem is:

$$V(0, W(0), Y_0) = \max_{c_{Y_t}(t)} \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) u(c_{Y_t}(t), q_{Y_t}(t)) dt \mid Y_0 \right] \quad (10)$$

$$s. t. W(0) = W_0,$$

$$W(t) \geq 0, W(T) = 0,$$

$$\frac{\partial W(t)}{\partial t} = rW(t) + m_{Y_t}(t) - c_{Y_t}(t)$$

As in the deterministic model presented in Section II.B, we focus on the non-trivial case where the non-negative wealth constraint does not bind. Define the consumer's objective function at time u as:

$$J(u, i) = \mathbb{E} \left[\int_0^{T-u} e^{-\rho t} \exp \left\{ - \int_0^t \mu(u+s) ds \right\} u(c_{Y_{u+t}}(u+t), q_{Y_{u+t}}(u+t)) dt \mid Y_u = i \right] \quad (11)$$

We can then write the objective function recursively as:

$$J(u, i) = \int_0^{T-u} e^{-\rho t} \exp \left\{ - \int_0^t \left(\bar{\mu}_i(u+s) + \sum_{j \neq i} \lambda_{ij}(u+s) \right) ds \right\} \left(u(c_i(u+t), q_i(u+t)) + \sum_{j \neq i} \lambda_{ij}(u+t) J(u+t, j) \right) dt$$

Define the optimal value function as

$$V(t, W(t), i) = \max_{c_i(s), s \geq t} \{J(t, i)\}$$

Under conventional regularity conditions, we know that if V and its partial derivatives are continuous, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) system of equations:

$$\begin{aligned} & (\rho + \bar{\mu}_i(t)) V(t, W(t), i) \\ & = \max_{c_i(t)} \left\{ u(c_i(t), q_i(t)) + \frac{\partial V(t, W(t), i)}{\partial W(t)} [rW(t) + m_i(t) - c_i(t)] \right. \\ & \quad \left. + \frac{\partial V(t, W(t), i)}{\partial t} + \sum_{j \neq i} \lambda_{ij}(t) [V(t, j) - V(t, i)] \right\}, i = 1, \dots, n \end{aligned} \quad (12)$$

We are interested in understanding how optimal consumption, and thus the value of life, changes over the life-cycle in this problem. We follow Pappas and Webster (2013), who demonstrate that it is possible to

⁸ Reichling and Smetters (2015) show that when annuity markets are incomplete, stochastic mortality and correlated medical costs can explain the puzzling observation that many households do not fully annuitize their wealth. They take the positive correlation between health shocks and medical spending as a given. Our study sheds light on *why* these two phenomena are positively correlated.

reformulate a stochastic optimization problem as a deterministic problem that takes $V(t, W(t), j), j \neq i$, as exogenous. This then allows us to apply the Pontryagin maximum principle and derive analytic expressions.

Lemma 1:

The optimal value function for $Y_0 = i, V(0, W(0), i)$, for the following deterministic optimization problem also satisfies the HJB given by (12), for each $i \in \{1, \dots, n\}$:

$$V(0, W_0, i) = \max_{c_i(t)} \left[\int_0^T e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) dt \right] \quad (13)$$

$$s. t. \frac{\partial W(t)}{\partial t} = rW(t) + m_i(t) - c_i(t)$$

where $V(t, W(t), j)$ are taken as exogenous.

Proof of Lemma 1: see Appendix A

Following Bertsekas (2005), the present value Hamiltonian corresponding to (13) is

$$H(W(t), c_i(t), p_t^{(i)}) = e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) + p_t^{(i)} [rW(t) - c_i(t) + m_i(t)]$$

where $p_t^{(i)}$ is the costate variable for state i . The necessary costate equation is

$$\dot{p}_t^{(i)} = -p_t^{(i)} r - e^{-\rho t} \tilde{S}(i, t) \sum_{j \neq i} \lambda_{ij}(t) \frac{\partial V(t, W(t), j)}{\partial W(t)}$$

The solution to the costate equation can be obtained using the variation of the constant method:

$$p_t^{(i)} = \left[\int_t^T e^{(r-\rho)s} \tilde{S}(i, s) \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds \right] e^{-rt} + \theta^{(i)} e^{-rt}$$

where $\theta^{(i)}$ is a constant. The necessary first-order condition for consumption is:

$$p_t^{(i)} = e^{-\rho t} \tilde{S}(i, t) u_c(c_i(t), q_i(t)) \quad (14)$$

where the marginal utility of wealth at time $t = 0$ is $\frac{\partial V(0, W_0, i)}{\partial W_0} = p_0^{(i)} = u_c(c_i(0), q_i(0))$. Since the Hamiltonian is concave in c and linear in W , the necessary conditions for optimality are also sufficient (Seierstad and Sydsaeter 1977).

To analyze the value of life, we let $\delta(t)$ be a perturbation on the mortality rate in state i with $\int_0^T \delta(t) dt = 1$ and consider

$$\tilde{S}^\varepsilon(i, t) = \exp \left[- \int_0^t (\bar{\mu}_i(s) - \varepsilon \delta(s)) + \sum_{j \neq i} \lambda_{ij}(s) ds \right], \text{ where } \varepsilon > 0$$

We first derive an expression for the effect of this perturbation on expected lifetime utility.

Lemma 2:

The marginal utility of life extension in state i is equal to:

$$\left. \frac{\partial V}{\partial \varepsilon} \right|_{\varepsilon=0} = \int_0^T \left[e^{-\rho t} \left(\int_0^t \delta(s) ds \right) \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) \right] dt$$

Proof of Lemma 2:

From (13), the marginal utility of life-extension is

$$\begin{aligned} \left. \frac{\partial V}{\partial \varepsilon} \right|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} \exp \left\{ - \int_0^t (\mu(s) - \varepsilon \delta(s)) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} \left(u(c_i^\varepsilon(t), q_i(t)) \right. \\ &\quad \left. + \sum_{j \neq i} \lambda_{ij}(t) V(t, W^\varepsilon(t), j) \right) dt \Big|_{\varepsilon=0} \\ &= \int_0^T e^{-\rho t} \left(\int_0^t \delta(s) ds \right) \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) dt \\ &\quad + \int_0^T e^{-\rho t} \tilde{S}(i, t) \left(u_c(c_i(t), q_i(t)) \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} + \sum_{j \neq i} \lambda_{ij}(t) \frac{\partial V(t, W(t), j)}{\partial W} \frac{\partial W^\varepsilon(t)}{\partial \varepsilon} \right) dt \Big|_{\varepsilon=0} \end{aligned}$$

where $c_i^\varepsilon(t)$ and $W^\varepsilon(t)$ represent the equilibrium variations in $c_i(t)$ and $W(t)$ caused by this perturbation. We conclude the proof by showing that the second term in the last equality is equal to 0. Note that along this path, wealth at time t is equal to

$$W(t) = W_0 e^{rt} + \int_0^t e^{r(t-s)} m_i(s) ds - \int_0^t e^{r(t-s)} c_i(s) ds,$$

which implies $\frac{\partial W^\varepsilon(t)}{\partial \varepsilon} = - \int_0^t e^{r(t-s)} \frac{\partial c_i^\varepsilon(s)}{\partial \varepsilon} ds$. From the solution to the costate equation, we know that

$$e^{-\rho t} \tilde{S}(i, t) u_c(c_i(t), q_i(t)) = \left[\int_t^T e^{(r-\rho)s} \tilde{S}(i, s) \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds \right] e^{-rt} + \theta^{(i)} e^{-rt}$$

Thus, we can rewrite the second term in the expression for $\left. \frac{\partial V}{\partial \varepsilon} \right|_{\varepsilon=0}$ above as

$$\begin{aligned} &\int_0^T \left[\int_t^T e^{(r-\rho)s} \tilde{S}(i, s) \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds + \theta^{(i)} \right] e^{-rt} \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} dt \\ &\quad - \int_0^T e^{-\rho t} \tilde{S}(i, t) \sum_{j \neq i} \lambda_{ij}(t) \frac{\partial V(t, W(t), j)}{\partial W} \int_0^t e^{r(t-s)} \frac{\partial c_i^\varepsilon(s)}{\partial \varepsilon} ds dt \Big|_{\varepsilon=0} \end{aligned}$$

$$\begin{aligned}
&= \int_0^T \left[\int_t^T e^{(r-\rho)s} \tilde{S}(i, s) \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds \right] e^{-rt} \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} dt \\
&\quad - \int_0^T \left[\int_t^T e^{(r-\rho)s} \tilde{S}(i, s) \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds \right] e^{-rt} \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} dt + \int_0^T \theta^{(i)} e^{-rt} \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} dt \Bigg|_{\varepsilon=0} \\
&= \theta^{(i)} \frac{\partial}{\partial \varepsilon} \underbrace{\int_0^T e^{-rt} c_i^\varepsilon(t) dt}_{W_0 + \int_0^T e^{-rt} m_i(t) dt} \Bigg|_{\varepsilon=0} \\
&= 0
\end{aligned}$$

QED

In order to facilitate comparison to the deterministic case, it is useful to derive an expression for the marginal utility of wealth at time t .

Lemma 3:

The expected marginal utility of wealth in state i at time t is equal to:

$$\frac{\partial V(t, W(t), i)}{\partial W(t)} = u_c(c_i(t), q_i(t)) = \mathbb{E} \left[e^{(r-\rho)(\tau-t)} \exp \left\{ - \int_t^\tau \mu(s) ds \right\} u_c(c_{Y_\tau}(\tau), q_{Y_\tau}(\tau)) \Big| Y_t = i \right]$$

Proof of Lemma 3: see Appendix A

Our next result demonstrates that the value of statistical life takes the same basic form as in the deterministic case.

Proposition 4:

Choosing once again the Dirac delta function for $\delta(\cdot)$ simplifies the expression for the marginal utility of life-extension:

$$\begin{aligned}
\frac{\partial V}{\partial \varepsilon} \Big|_{\varepsilon=0} &= \int_0^T \left[e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) \right] dt \\
&= \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) u(c_{y_t}(t), q_{y_t}(t)) dt \Big| Y_0 = i \right]
\end{aligned}$$

Dividing the result by the marginal utility of wealth at time $t = 0$ and then applying **Lemma 3** shows that the value of statistical life takes the same basic form as in the deterministic case:

$$VSL(i) = \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) \frac{u(c_{y_t}(t), q_{y_t}(t))}{u_c(c_{Y_0}(0), q_{Y_0}(0))} dt \Big| Y_0 = i \right] = \int_0^T e^{-rt} v(i, t) dt \quad (15)$$

where the value of a statistical life-year is equal to the expected utility of consumption normalized by the expected marginal utility of consumption:

$$v(i, t) = \frac{\mathbb{E} \left[S(t) u(c_{y_t}(t), q_{y_t}(t)) \Big| Y_0 = i \right]}{\mathbb{E} \left[S(t) u_c(c_{y_t}(t), q_{y_t}(t)) \Big| Y_0 = i \right]}$$

Proof of Proposition 4: see Appendix A

As before, the value of statistical life is proportional to the expected discounted (lifetime) utility of consumption, and inversely proportional to the marginal utility of consumption. As we shall show below, a negative health shock increases current consumption, causing the net effect on VSL to be ambiguous. This parallels the result we showed previously that removing access to annuitization, thereby exposing a consumer to mortality risk, has an ambiguous effect on VSL.

We can derive an expression for the life-cycle profile of consumption from (14), the first-order condition for p_t . Differentiating with respect to t , plugging in the result for the costate equation and its solution, and rearranging yields

$$\frac{\dot{c}_i}{c_i} = \sigma(r - \rho) + \sigma\eta\frac{\dot{q}}{q} - \sigma\bar{\mu}_i(t) - \sigma \sum_{j \neq i} \lambda_{ij}(t) \left[1 - \frac{u_c(c_j(t), q_j(t))}{u_c(c_i(t), q_i(t))} \right] \quad (16)$$

As in the deterministic case, the rate of change is a declining function of the individual's current mortality rate, $\bar{\mu}_i(t)$: removing the annuity market “pulls consumption earlier” in the life-cycle. Unlike in the deterministic case, there is now an additional source of mortality risk, captured by the fourth term in equation (16). This term represents the possibility that the consumer might transition to a different health state in the future, and shifts consumption further still if the consumer is likely to fall ill in the future.

We caution that equation (16) is specific to an individual's health state i and is not easily aggregated across health states. That is, one cannot infer from equation (16) whether stochastic mortality *on average* causes consumption to shift forward relative to deterministic mortality. In general, one should expect stochastic mortality to shift consumption forward by less than in the deterministic case. Intuitively, this is because a stochastic environment allows an individual to react to unanticipated health shocks by adjusting her consumption. Put differently, a deterministic model is equivalent to a stochastic model where the consumer is forced to keep consumption constant across states. Consumers prefer the ability to adjust consumption, so that they can consume less in healthy states and more in sick states. Our numerical exercises, which assume CRRA utility, find that on net stochastic mortality causes consumers to shift consumption forward a bit less than deterministic mortality.

What happens when an individual transitions to a new health state? Because the consumer is not insured against mortality or quality of life risks, consumption will jump. The sign of the jump can be positive or negative, depending on the characteristics of the new health state relative to the old state. Because there is no consensus regarding the sign of health state dependence ($u_{cq}(\cdot)$), let alone the magnitude, we hold quality of life constant for the time being, and return to this issue in our empirical analysis.⁹ Focusing on mortality, the model predicts that transitioning to a state where the current mortality and future expected mortality are high will shift consumption forward (see Figure 2), and vice versa. Our next result proves this formally for a two-state case.¹⁰

⁹ Finkelstein et al. (2013), Sloan et al. (1998), and Viscusi and Evans (1990) find evidence of negative state dependence. Edwards (2008) and Lillard and Weiss (1988) find evidence of positive state dependence. Evans and Viscusi (1991) find no evidence of state dependence. Murphy and Topel (2006) assume negative state dependence when performing their calibration exercises, while Hall and Jones (2007) assume state independence.

¹⁰ The proof can be extended to allow for a larger number of states, but the conditions required to sign the jump in consumption then become a complicated function of the matrix of transition probabilities and state-specific mortality rates. The two-state case conveys the basic result without a meaningful loss of generality.

Proposition 5:

Let there be $n = 2$ states with identical quality of life profiles, so that $q_1(s) = q_2(s) \forall s$. Assume that $\bar{\mu}_1(s) < \bar{\mu}_2(s) \forall s$, so that state 1 is “healthy” and state 2 is “sick.” Suppose that the consumer transitions from state 1 to state 2 at time t , with no change in income (i.e., $m_1(t) = m_2(t)$). Then $c_1(t) < c_2(t)$.

Proof of Proposition 5: see Appendix A

It follows immediately from **Proposition 5** that the value of near-term life-years will increase, and the value of distant life-years will decrease, when transitioning from a healthy state with low mortality to a sick state with higher mortality. Whether VSL rises or falls is ambiguous, however. A rise in mortality risk lowers lifetime utility, which reduces VSL, but it also reduces the marginal utility of consumption, which increases VSL. Thus the net effect depends on the curvature of the utility function relative to the curvature of the marginal utility function. The elasticity of intertemporal substitution, σ , is a common measure of the utility curvature. The analogous measure for the curvature of marginal utility is prudence (Kimball 1990). Define relative prudence as

$$\pi \equiv -\frac{cu_{ccc}(\cdot)}{u_{cc}(\cdot)}$$

Our next result provides a sufficient condition for VSL to rise following an adverse mortality shock.

Proposition 6:

Consider a two-state setting with assumptions set out in **Proposition 5**. Assume further that preferences satisfy the additional condition

$$\pi < \frac{2}{\sigma}$$

Suppose that the consumer transitions from state 1 to state 2 at time t , and that $\lambda_{12}(\tau) = 0 \forall \tau > t$. Then $VSL(1, t) < VSL(2, t)$.

Proof of Proposition 6: see Appendix A

The condition specified in **Proposition 6** is satisfied by many common preferences, such as CRRA with $\sigma < 1$ (which we employ in our numerical exercises) and quadratic preferences. Consumers with inelastic demand, i.e., preferences with a low value for σ , find it costly to reallocate consumption over time. They therefore have a high willingness-to-pay for life-extension and are more likely to exhibit a rise in VSL following an adverse mortality shock. Likewise, consumers with low levels of prudence have nearly-linear marginal utility that decreases rapidly with consumption. This generates a high willingness-to-pay for life-extension following a shock that increases consumption.

III.A. The value of statistical illness

Unlike the deterministic model, the stochastic model permits an investigation not only into the value of preventing death, but also into the value of preventing transitions to other health states. This requires only a slight modification to the analysis prevented above, and will result in a more general concept we term the value of statistical illness. With a slight abuse of notation, let state $N + 1$ correspond to death, so that $V(t, W(t), N + 1) = 0$. Let $\delta_{ij}(t)$, $i, j \leq N$, be a perturbation on the transition intensity $\lambda_{ij}(t)$ and $\delta_{i, N+1}(t)$ be a perturbation on the mortality rate $\bar{\mu}_i(t)$, where $\sum_{j=1, j \neq i}^{N+1} \int_0^T \delta_{ij}(t) dt = 1$, and consider

$$\tilde{S}^\varepsilon(i, t) = \exp \left[- \int_0^t (\bar{\mu}_i(s) - \varepsilon \delta_{i, N+1}(t)) + \sum_{j=1, j \neq i}^N (\lambda_{ij}(s) - \varepsilon \delta_{ij}(t)) ds \right], \text{ where } \varepsilon > 0$$

Proposition 7:

The marginal utility of preventing an illness or death is given by:

$$\begin{aligned} \frac{\partial V}{\partial \varepsilon} \Big|_{\varepsilon=0} = \int_0^T e^{-\rho t} \tilde{S}(i, t) & \left[\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) \right. \\ & \left. - \sum_{j \neq i} \delta_{ij}(t) V(t, W(t), j) \right] dt \end{aligned}$$

Proof of Proposition 7:

From (13), the marginal utility of preventing an illness or death is defined as

$$\begin{aligned} \frac{\partial V}{\partial \varepsilon} \Big|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} \exp \left\{ - \int_0^t (\bar{\mu}_i(s) - \varepsilon \delta_{i, N+1}(t)) + \sum_{j \neq i} (\lambda_{ij}(s) - \varepsilon \delta_{ij}(s)) ds \right\} & \left(u(c_i^\varepsilon(t), q_i(t)) \right. \\ & \left. + \sum_{j \neq i} (\lambda_{ij}(t) - \varepsilon \delta_{ij}(t)) V(t, W^\varepsilon(t), j) \right) dt \Big|_{\varepsilon=0} \\ = \int_0^T e^{-\rho t} \tilde{S}(i, t) & \left[\left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) - \sum_{j \neq i} \delta_{ij}(t) V(t, W(t), j) \right] dt \\ & + \int_0^T e^{-\rho t} \tilde{S}(i, t) \left(u_c(c_i^\varepsilon(t), q_i(t)) \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} + \sum_{j \neq i} \lambda_{ij}(t) \frac{\partial V(t, W(t), j)}{\partial W} \frac{\partial W^\varepsilon(t, i)}{\partial \varepsilon} \right) dt \end{aligned}$$

Following the same argument as in the VSL case, the second term in the last equality is equal to 0.

QED

The value of preventing an illness or death is equal to the marginal rate of substitution between the transition perturbation and wealth:

$$\frac{\partial V / \partial \varepsilon}{\partial V / \partial W} = \int_0^T \frac{e^{-\rho t} \tilde{S}(i, t)}{u_c(c_i(0), q_i(0))} \left[\left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) - \sum_{j \neq i} \delta_{ij}(t) V(t, W(t), j) \right] dt$$

As before, it is helpful to choose the Dirac delta function for $\delta(\cdot)$, so that the probability is perturbed at $t = 0$ and remains unaffected otherwise. It is also helpful to consider a reduction in the transition probability for only one alternative state, j_0 , so that $\delta_{ij}(t) = 0 \forall j \neq j_0$. Applying these two conditions then yields what we term the value of statistical illness, $VSI(i, j)$:

$$\begin{aligned} VSI(i, j) &= \frac{V(0, W(0), i) - V(0, W(0), j)}{u_c(c_i(0), q_i(0))} \\ &= VSL(i) - VSL(j) \frac{u_c(c_j(0), q_j(0))}{u_c(c_i(0), q_i(0))} \end{aligned} \tag{17}$$

The interpretation of VSI is analogous to VSL: it is the amount that 1,000 individuals would collectively be willing to pay in order to eliminate a current disease risk that is expected to befall one of them. Note that if health state j corresponds to death, so that $VSL(j) = VSL(N + 1) = 0$, then $VSI(i, j) = VSL(i)$. Thus, VSI is a generalization of VSL.

It is instructive to compare (17) to the expression for VSI obtained when the consumer is fully annuitized (derivation available in Appendix C):

$$VSI^*(i, j) = VSL^*(i) - VSL^*(j) \quad (18)$$

Equation (18) provides justification for the common practice of equating the values of prevention and treatment.¹¹ Conventional cost-effectiveness analysis relies upon the standard fully annuitized framework that assumes the value of a life-year is equal across health states (holding quality of life constant). If the value of a life-year is constant, then equation (18) implies that prevention and treatment are equally valuable, as long as they add the same number of expected life-years. For example, conventional cost-effectiveness frameworks value a treatment that prevents the onset of an illness that lowers life expectancy by 10 years the same as a therapeutic treatment that cures an illness and adds 10 years of life expectancy (Drummond et al. 2005b). Equation (17) shows that removing access to annuity markets breaks this equivalence between treatment and prevention. VSI in this case is not equal to the simple difference in VSL between the healthy and sick states, because VSL in the sick state is valued from the perspective of the sick, who have a lower marginal utility of consumption due to a shorter life span.

To summarize, the stochastic mortality model yields the following implications:

- The values of near-term life-years rise, and distant life-years fall, when an individual transitions to a higher mortality state;
- The value of statistical life may rise or fall when an individual transitions to a higher mortality state; if the individual is insufficiently prudent, then it will rise
- Therapies that increase survival by treating sick patients are not the same as, and may even be more valuable than, those that add the same amount of life expectancy by preventing illness in healthy patients.

IV. ESTIMATES OF THE VALUE OF LIFE

IV.A. Framework

We will work with the discrete time analogue of our model. There are n health states. Denote the transition probabilities between health states by:

$$p_{ij}(t) = \mathbb{P}[Y_{t+1} = j | Y_t = i]$$

As in the continuous time model, the mortality rate at time t depends on the individual's health state:

$$d_t = \sum_{j=1}^n \bar{d}_t^j \mathbf{1}\{Y_t = j\}$$

¹¹ When the consumer is fully annuitized, the value of her annuity depends on her health state. In particular, if she purchases an annuity in state i and then later transitions to a worse health state j , causing her life expectancy to fall, then the value of her annuity will also fall. This technicality is not reflected in the notation for equation (18); see Appendix C for details and discussion.

where $\{\bar{d}_t^j\}$ are given and $\mathbf{1}\{Y_t = j\}$ is an indicator variable equal to 1 if the individual is in state j at time t and 0 otherwise. The probability of surviving from time period t to time period s is denoted as $S_t(s)$, where

$$S_t(t) = 1,$$

$$S_t(s) = S_t(s-1)(1 - d_{s-1}), s > t$$

Let c_t , q_t , and w_t denote consumption, quality of life, and wealth in period t , respectively. Let ρ denote the utility discount rate, and r the interest rate. Assume that in each period the consumer receives an exogenously determined income, y_t , and that the maximum lifespan of a consumer is T (i.e., $d_T = 1$). Our baseline model assumes there is no bequest motive, although we plan to relax this assumption in a later exercise.

The consumer's maximization problem is

$$\max_{\{c_t\}} \mathbb{E}_0 \left[\sum_{t=0}^T e^{-\rho t} S_0(t) u(c_t, q_t) \right]$$

subject to

$$w_0 \text{ given,}$$

$$w_t \geq 0,$$

$$w_{t+1} = (w_t + y_t - c_t)e^r$$

We assume throughout that $r = \rho = 0.03$ (Siegel 1992; Moore and Viscusi 1990). Finally, we assume that utility takes the following CRRA form:

$$u(c, q) = q \frac{c^{1-\gamma}}{1-\gamma} - \frac{\underline{c}^{1-\gamma}}{1-\gamma} \quad (19)$$

As discussed in Section III, there is no consensus regarding the sign or magnitude of health state dependence ($u_{cq}(\cdot)$). Here, we assume a multiplicative relationship, so that the marginal utility of consumption is higher when quality of life is high, and vice versa. Because a negative health shock is likely to reduce both life expectancy and quality of life, this assumption is conservative in the following sense: an individual with these preferences is less likely to subsequently exhibit an increase in consumption, or VSL, following a negative health shock, as compared to an individual with state independent preferences.

We have normalized the utility of death to zero in (19). The consumer receives positive utility if she consumes an amount greater than \underline{c} , which represents a subsistence level of consumption. Consuming an amount less than \underline{c} generates utility that is worse than death. Although adding a constant to the utility function has no effect in most settings, Rosen (1988) explains how the level of utility matters when valuing life extension and notes that well-behaved preferences requires that utility be positive. We are unaware of any empirical evidence on the magnitude of \underline{c} , the subsistence level of consumption in the United States. We assume it is equal to \$5,000.

The parameter γ is the inverse of the elasticity of intertemporal substitution, an important determinant of the value of life and the value of annuitization. We follow Hall and Jones (2007) and set $\gamma = 2$ in our

analyses. As points of reference, Murphy and Topel (2006) set $\gamma = 1.25$ while Brown (2001) uses survey data to estimate a mean value of $\gamma = 3.95$.

We employ dynamic programming techniques to solve for the optimal consumption path (see Appendix for details). The value function is defined as:

$$V(t, w_t, i) = \max_{\{c_t\}} \mathbb{E} \left[\sum_{s=t}^T e^{-\rho(s-t)} S_t(s) u(c_s) \middle| Y_t = i \right]$$

We can use the value function to rewrite the optimization problem as a recursive Bellman equation:

$$V(t, w_t, i) = \max_{\{c_t\}} \left[u(c_t) + \frac{1 - d_t}{e^\rho} \sum_{j=1}^N p_{ij}(t) V(t+1, (w_t + y_t - c_t)e^r, j) \right]$$

Once we have solved for the optimal consumption path, we can use the analytical formulas derived in the previous section to calculate the value of life.

We are aware that there is significant uncertainty among economists regarding the proper values of many of the parameters in our model. Our goal in this section is to illustrate the significance of our insights when our model is applied to real-world data using reasonable parameterizations.

IV.B. Annuitization, retirement policy, and the value of life

In this section we explore the link between annuitization and retirement policy. We build up to these results by calculating how the value of statistical life varies over the life-cycle under alternative annuitization policies. We then calculate how these alternative policies influence the value of permanent reductions in mortality. All our calculations account for the effect of mortality reduction on net savings, regardless of the degree of annuitization, in order to facilitate comparison across different annuitization scenarios and because this is appropriate when estimating the social value of increased longevity. (See footnote 5.)

We initiate the model at age 20 and assume nobody survives past age 100. We use data on age-specific mortality rates from www.mortality.org. Because these mortality data are not available by health state, in this section we will work with a deterministic model. (This corresponds to specifying $n = 1$ health states in the framework above.) We also abstract from the role of quality of life and set $q_t = 1 \forall t$ because aggregate, nationally representative data on quality-of-life trends are not generally available. We choose the individual's labor earnings, $\{m_t\}$, to fit data on life-cycle earnings as estimated by the Current Population Survey and the Health and Retirement Survey. See Appendix B1 for details.

The individual's period income is $y_t = (1 - \tau)m_t + a_t$, where a_t is nonwage defined-benefit income financed by an earnings tax, τ . We consider three different policy scenarios in the main text. In the first, financial markets are absent and the consumer's income corresponds to labor earnings: $y_t^1 = m_t$. The second scenario introduces an actuarially fair Social Security program that provides an annuity equal to \$16,195 beginning at age 65.¹² Here, the consumer is partially annuitized, but she lacks access to financial markets and cannot borrow against her future income. Thus, her consumption is limited by current period income and savings from prior periods. The third scenario increases the size of the Social Security

¹² This corresponds to the average retirement benefit paid by Social Security to retired workers in 2016 (www.ssa.gov/policy/docs/quickfacts/stat_snapshot/2016-07.pdf).

pension by 50 percent. For interested readers, the appendix presents results for the case where the consumer fully annuitizes at age 20 and enjoys a constant annuity stream, $\bar{y} = \bar{a}$, provided by an actuarially fair and complete annuities market. The income streams in all scenarios are related according to the following equation:

$$\sum_{t=1}^T \frac{y_t^1 S_t}{e^{r(t-1)}} = \sum_{t=1}^T \frac{y_t^2 S_t}{e^{r(t-1)}} = \sum_{t=1}^T \frac{y_t^3 S_t}{e^{r(t-1)}} = \bar{y} \sum_{t=1}^T \frac{S_t}{e^{r(t-1)}}$$

Our assumed interest rate of 3 percent and our data on mortality and earnings imply a full annuity value of $\bar{y} = \$38,019$.

The life-cycle profiles of consumption for the first two scenarios are displayed in Figure 3. Consumption is constrained by the consumer's low income in early life. She saves during middle age when income is high, and then consumes her savings during retirement until eventually her consumption equals her pension (if available). Consumption for an individual with no Social Security pension is "shifted forward" relative to an individual with a pension. This effect is particularly dramatic in the final 10 years of life, when old consumers outlive their wealth.¹³ This is not surprising: a primary benefit of an annuity is its ability to provide income to consumers in their oldest ages.

Figure 4 shows that this difference in consumption causes a corresponding difference in the value of a life-year (VLY). Individuals place a low value on VLY in early and late ages, when consumption is low. The drop at age 65 reflects the effect of retirement on the net savings component of the value of life.

Figure 5 displays the corresponding value of statistical life (VSL) for these two scenarios, as calculated by equation (7). At age 40, VSL is equal to \$7 million and \$8 million for individuals with no pensions and with Social Security, respectively. These values are within the ranges estimated by empirical studies of VSL for working-age individuals (Viscusi and Aldy 2003). Figure 5 also reveals that VSL is greater at older ages for individuals with retirement annuities than it is for individuals with no annuities. This suggests that public annuity programs are complementary with retiree healthcare programs and other investments in life-extension for the elderly population.

Finally, we calculate the value of historical reductions in mortality for these different annuitization scenarios, as well as the prospective value of permanent reductions in future mortality for selected diseases. Let the mortality reduction for age a be denoted as $\delta(a)$, and let δ denote a vector of mortality reductions for all ages. As in Murphy and Topel (2006), we calculate the total social value of a mortality reduction by aggregating over the age distribution of the 2015 U.S. population

$$Social\ Value = \sum_{a=0}^{110} VLE(a, \delta) f(a)$$

¹³ Hubbard, Skinner, and Zeldes (1995) show that failing to include a "welfare floor" in the budget constraint causes life-cycle models to overestimate savings for low-income households. Our calibration exercises model median-income individuals, however, for whom this issue is less important.

where $VLE(a, \delta)$ is defined as in equation (5), and $f(a)$ is the count of individuals alive in 2015 at age a .¹⁴

We report our results in Table 1. Like Murphy and Topel (2006), we find that the social value of past longevity gains are enormous: the post-1940 gains are worth over \$100 trillion today, and the post-1970 gains are worth over \$50 trillion. Comparing results for different annuitization scenarios informs our understanding of the interaction between retirement policies and the value of longevity. For example, consider the introduction of Social Security over the last century. Comparing Column (1) to Column (2) of Table 1 suggests that this increased the value of post-1940 longevity gains by \$11.5 trillion (10.6 percent), and increased the value of post-1970 gains by \$6.2 trillion (11.6 percent). One way to interpret these values is to compare them to the current aggregate insurance value of Social Security, which is approximately \$24 trillion (Mitchell et al. 1999). Thus, incorporating the value of post-1940 health gains raises the social welfare value of Social Security by nearly 50 percent.

Table 1 also reveals that Social Security has raised the value of a 10 percent cancer mortality reduction by \$446 billion, or 13 percent. Alternatively, it has raised the value of a 10 percent reduction in all-cause mortality by \$1.43 trillion (12 percent). Increasing the size of Social Security pensions by 50 percent would add \$488 billion more to that value.

More generally, the shift towards publicly and privately funded retirement plans has raised the value of life at older ages, and thus may have contributed to the observed increase in demand for elderly healthcare. This dovetails with the point, made by Philipson and Becker (1998), that the moral hazard effects of retirement programs also increase the willingness to pay for longevity. It is therefore not surprising that public spending on healthcare – particularly for the elderly – has grown enormously in developed countries.

Our model predicts that annuitization raises VSL for the elderly. This should cause them to spend more on healthcare and invest more in healthy behaviors, which in turn should ultimately manifest in increased life expectancy. Philipson and Becker (1998) analyze data from Virga (1996) to show that people with more generous annuities live longer than those with less generous annuities. They interpret this as the effect of endogenous longevity investments, which are encouraged among highly annuitized individuals. An additional explanation is that annuitization increases the value of statistical life, as we emphasize here. These are compatible and consistent interpretations.

IV.C. Heterogeneity in VSL

The conventional economic theory of life extension conceives of VSL as depending primarily on age and consumption. The general framework with stochastic mortality and incomplete annuitization implies instead a substantial amount of variability in VSL within these categories alone. Individuals who have experienced a recent negative mortality shock have systematically higher VSL, but this VSL premium decays over time. We use real-world data on mortality, mortality shocks, and income to estimate the degree to which VSL varies within the traditional categories, and the factors explaining the variation.

We use data on mortality and mortality shocks from the Future Elderly Model (FEM), a widely published microsimulation model that employs nationally representative data from the Health and Retirement Study (Michaud et al. 2011; Goldman et al. 2005; Lakdawalla, Goldman, and Shang 2005; Goldman et al. 2009;

¹⁴ We assume $VLE(a, \delta) = VLE(20, \delta)$ for $a < 20$, and equal to $VLE(100, \delta)$ for $a > 100$. Unlike Murphy and Topel (2006), our social value calculation does not account for the value that mortality reductions generate for future (unborn) populations.

Lakdawalla et al. 2009; Goldman et al. 2013; Michaud et al. 2012; Goldman et al. 2010). The FEM uses real-world risks of disease incidence, and mortality rates by disease state, in order to estimate longevity for people over the age of 50 with different comorbid conditions.¹⁵ The FEM accounts for six different chronic conditions (cancer, diabetes, heart disease, hypertension, chronic lung disease, and stroke) and six different impaired activities of daily living (bathing, eating, dressing, walking, getting in or out of bed, and using the toilet). This is quite useful for our current purposes, because it provides us with an empirically relevant set of estimates for what mortality risk looks like under different disease states.

We divide the health space within the FEM into 20 states. Each state corresponds to the number (0, 1, 2, 3 or more) of impaired activities of daily living (ADL) and the number (0, 1, 2, 3, 4 or more) of chronic conditions, for a total of $4 \times 5 = 20$ health states. States are ordered first by number of ADL's and then by number of chronic diseases. So state 1 corresponds to 0 ADL's and 0 chronic conditions, state 2 corresponds to 0 ADL's and 1 chronic condition, and so on. For each health state and age, the FEM estimates the probability of dying, and the probability of transitioning to each of the other health states in the next year. As in the theoretical model, individuals can transition only to higher-numbered states, i.e., $p_{ij}(t) = 0 \forall j < i$. In other words, all ADL's and chronic conditions are permanent.

We will be conditioning on income and wealth, and frequently reporting results by age, to eliminate all the variation that the conventional model of VSL would imply. Table 2 presents basic descriptive statistics for the data provided by the FEM model. Life expectancy at age 50 ranges from 35 years for a healthy individual in state 1 to 15 years for an ill individual in state 20. Columns (6)-(8) of Table 2 report the probability that an individual exits their health state after one year, i.e., acquires at least one new ADL or chronic condition. Health states are relatively persistent, with exit rates never exceeding 30%. State 20 is an absorbing state with an exit rate of 0 percent.

We make two simplifying assumptions that allow us to generate exact, analytical solutions to the stochastic mortality model: we assume an individual can borrow against her future income, and that income is not survival contingent. These two assumptions imply an equivalence between income and wealth, allowing us to ignore income and to work with wealth only.¹⁶ See the appendix for details on how the model is calculated. We set initial wealth equal to \$735,170 at age 50, which corresponds to wealth at age 50 as estimated by the deterministic model presented in the prior section. All other assumptions are the same as before.

Figure 6 displays life-cycle profiles of consumption and VSL for an individual who never develops a chronic condition or an ADL. Because she never suffers any health shocks, consumption and VSL decline smoothly over time. VSL at age 50 is equal to \$3.5 million and has the usual interpretation: it is the value that 1,000 individuals would collectively be willing to pay in order to reduce their individual risk of death by 1/1,000.

Figure 7 demonstrates a key mechanism for variability in VSL: The arrival of a health shock can increase VSL, sometimes substantially. The figure displays contrasting plots for an initially healthy individual who develops one ADL at age 60, and then a second ADL plus two chronic conditions at age 80. The first shock reduces her life expectancy by 4.1 years. The second one reduces her life expectancy by 7.6 years.

¹⁵ A description of its methodology is available at healthpolicy.app.box.com/FEMTechdoc.

¹⁶ Generalizing the model to allow for partial annuitization, bequest motives, and other similar modifications is possible but prohibits the calculation of an exact solution. We probe the sensitivity of our results to these alternatives with the deterministic mortality model.

In contrast to the healthy consumer, the sick consumer's consumption exhibits discontinuous jumps at age 60 and 80 as a result of the negative health shocks. The first shock is not large enough to stave off the declining trend in VSL, but the second one increases VSL at age 80 by over 50 percent, from \$1 million to \$1.5 million.

The arrival of shocks at the individual level in turn generate substantial variability in VSL in the aggregate. Figure 8 shows VSL at age 50 for individuals in one of the twenty different possible health states, as a function of life expectancy in each of those states. It ranges from \$3.5 million to \$4.5 million. This understates the total amount of heterogeneity that will occur, because shocks after age 50 will cause individuals' consumption paths to diverge even further. For example, at age 51 there are 210 ($= 20 \times 21/2$) different possible values for VSL.

IV.D. The value of prevention versus treatment

The stochastic mortality approach also allows us to calculate the value of preventing illness, which we defined as the value of a statistical illness (VSI). Figure 9 plots VSI at age 50 from the perspective of a healthy individual. Each point represents the healthy individual's willingness to pay for a marginal reduction in the probability of developing an illness corresponding to one of the 19 other health states. The values are increasing functions of life expectancy in the sick state because it is more valuable to prevent the onset of a lethal disease than a mild one. The highest VSI value is \$1.5 million, which corresponds to preventing the onset of a sick state with 3 ADL's and 4 chronic conditions. The interpretation of this value is analogous to VSL: it is the amount that 1,000 individuals would collectively be willing to pay in order to reduce their risk of developing this illness by 1/1000. In our framework, VSL can be interpreted as the willingness to pay to avoid the "illness" of dying, which correspond to a state with 0 years of remaining life expectancy. VSI corresponds to the willingness to pay to avoid a lower life expectancy, but still living, state.

How does the value of prevention compare to the value of treatment? We investigate this question by normalizing VSL and VSI by the number of life-year's "saved." We report the results of those calculations in Table 3. We normalize by discounted life expectancy rather than just life expectancy in order to facilitate comparison to the large literature on cost-effectiveness. Our VSL estimate implies that an individual with one chronic condition and no ADL's (health state 2) has a marginal willingness-to-pay of \$187,000 per life-year for a treatment that extends her life. By contrast, our VSI estimate implies that a healthy individual (health state 1) is only willing to pay \$147,000 per life-year for a preventive treatment that reduces her chances of developing 1 chronic condition. The ratio of these values is equal to 1.28. Column (7) of Table 2 shows that this ratio is always greater than 1, and sometimes significantly so: therapeutic treatments for very sick individuals with 3+ ADL's and 4+ chronic conditions are worth 2.6 times more than equally effective preventive care for healthy individuals. If we focus on the interquartile range, the values reported in Table 3 suggest that therapies are 50 to 100 percent more valuable than preventive treatments. This is in stark contrast to the standard cost-effectiveness framework, which values prevention and treatment equally. The model developed earlier provides the intuition behind these differences: all else equal, the marginal utility of consumption is higher for a healthy individual than for someone who has just suffered a health shock that reduces her lifespan. This in turn drives the difference in willingness-to-pay.

V. CONCLUSION

The economic theory surrounding the value of life has many important applications. Yet, like most theories, it suffers from a few anomalies that appear at odds with intuition, common sense, or empirical facts. We have demonstrated that several of these anomalies are easily explained without abandoning the standard framework, simply by relaxing its strong assumptions around the completeness of annuity

markets. Moreover, relaxing this assumption generates new predictions with implications for health policy and behavior. We show that VSL varies with the arrival of mortality shocks and with remaining life expectancy. A given gain in longevity is more valuable to a consumer who has less life remaining, and vice-versa. Even holding wealth and income fixed, VSL may vary by \$1 million or more for a 50-year-old. In addition, we demonstrate an interaction between annuity policy and health policy: Completing the annuity market may significantly increase the value of life-extension, especially for the elderly. For instance, the US Social Security program has increased the value of mortality reductions, adding as much as \$150 billion to the value of a 1 percent mortality decline.

Our findings have several implications for the valuation of health investments and for policy more generally. The value of a life-year will tend to vary across types of risk, not just across types of people. It can be more valuable to add one month of life for a patient facing a highly fatal disease than for one facing a much milder ailment. Thus, health spending should be more targeted towards the severely ill than current economic models of cost-effectiveness suggest.

In addition, public programs that expand the market for annuities might simultaneously boost the demand for life-extending technologies. Intuitively, annuities calm consumer fears about outliving their wealth and thus enable more aggressive investments in life-extension. Viewed differently, our results also show that market failures in annuities affect the value of statistical life, and thus the socially optimal level of health care spending.

Finally, our framework offers a single unified framework for valuing both life-extension and the prevention of illness. This provides a more practical tool for policymakers and decision makers, since many health investments involve preventing the deterioration of health, not a direct and immediate mortality risk.

Our analysis raises a number of important questions for further research. First, how does the value of longevity vary with endogenous demand for quality of life? Elsewhere, we have studied how incomplete health insurance enhances the value of medical technology that improves quality of life, because such technology acts as insurance by compressing the difference in utility between the sick and healthy states (Lakdawalla, Malani, and Reif 2017). Less clear is how demands for the quantity and quality of life interact with financial market incompleteness of various kinds. Second, what does the generalized value of life model mean for the value of different kinds of medical technologies? For instance, the model suggests that short-term survival gains for high-risk diseases are more valuable than previously believed, but very long-term survival gains might actually be less valuable than previously believed. Finally, what are the implications for the empirical literature on the value of statistical life? Empirical analysis has typically proceeded under the assumption that different kinds of mortality risk are all valued the same way, as long as they imply similar changes in the probability of dying (Viscusi and Aldy 2003; Hirth et al. 2000; Mrozek and Taylor 2002). Our framework casts doubt on this assumption and suggests the need for a more nuanced empirical approach. This missing insight may be one reason for the widely disparate estimates in the empirical literature on the value of a statistical life.

VI. REFERENCES

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VII. TABLES AND FIGURES

Table 1. Aggregate social value of historical and prospective reductions in mortality (billions of dollars)

	(1)	(2)	(3)
	No annuity	Social Security	Social Security + 50%
<u>Historical reduction:</u>			
1940-2010	\$109,252	\$120,799	\$126,398
1970-2010	\$53,447	\$59,651	\$62,728
<u>10% reduction, all ages:</u>			
All causes	\$11,715	\$13,142	\$13,630
Cancer	\$3,417	\$3,863	\$3,790
Diabetes	\$375	\$422	\$421
Heart disease	\$2,444	\$2,772	\$2,782
Homicide	\$109	\$106	\$180
Infectious diseases	\$165	\$188	\$192

Notes: Aggregate values were calculated by using the 2015 U.S. population by age. Column (1) presents estimates under the assumption that individuals have no annuities in retirement. Column (2) presents estimates under the assumption that individuals receive typical Social Security benefits that are financed by an earnings tax. Column (3) increases the generosity of Social Security by 50%, financed by an increase in the earnings tax. The net present value of individuals' wealth at age 20 is the same across all three columns.

Table 2. Descriptive statistics for the twenty health states in the stochastic mortality model

Health state	(1)	(2)	(3) Life expectancy		(5) Quality of life		(7) Exit probability		(8)
	ADL's	Chronic conditions	Age 50	Age 70	Age 50	Age 70	Age 50	Age 70	
1	0	0	32.8	16.3	0.884	0.874	4.3%	12.6%	
2	0	1	30.0	14.6	0.850	0.841	3.6%	10.5%	
3	0	2	26.3	12.5	0.812	0.804	3.6%	10.2%	
4	0	3	22.1	10.5	0.773	0.764	4.0%	10.3%	
5	0	4+	17.0	8.5	0.730	0.719	4.4%	7.6%	
6	1	0	28.4	14.2	0.830	0.816	6.4%	15.1%	
7	1	1	25.8	12.8	0.795	0.782	5.9%	12.8%	
8	1	2	22.0	11.0	0.754	0.745	5.6%	12.2%	
9	1	3	17.8	9.1	0.716	0.705	6.5%	11.8%	
10	1	4+	14.3	7.4	0.669	0.661	6.5%	8.7%	
11	2	0	26.1	13.2	0.781	0.763	7.5%	16.0%	
12	2	1	23.2	11.6	0.746	0.731	7.2%	13.9%	
13	2	2	19.5	10.0	0.706	0.693	7.6%	13.8%	
14	2	3	15.9	8.3	0.669	0.655	7.4%	12.9%	
15	2	4+	12.3	6.5	0.630	0.606	8.0%	11.4%	
16	3+	0	23.4	11.1	0.700	0.693	3.6%	10.4%	
17	3+	1	20.5	10.1	0.664	0.659	3.2%	8.8%	
18	3+	2	16.9	8.4	0.622	0.622	2.3%	7.2%	
19	3+	3	13.2	6.9	0.584	0.585	1.3%	5.2%	
20	3+	4+	9.6	5.3	0.536	0.540	0.0%	0.0%	

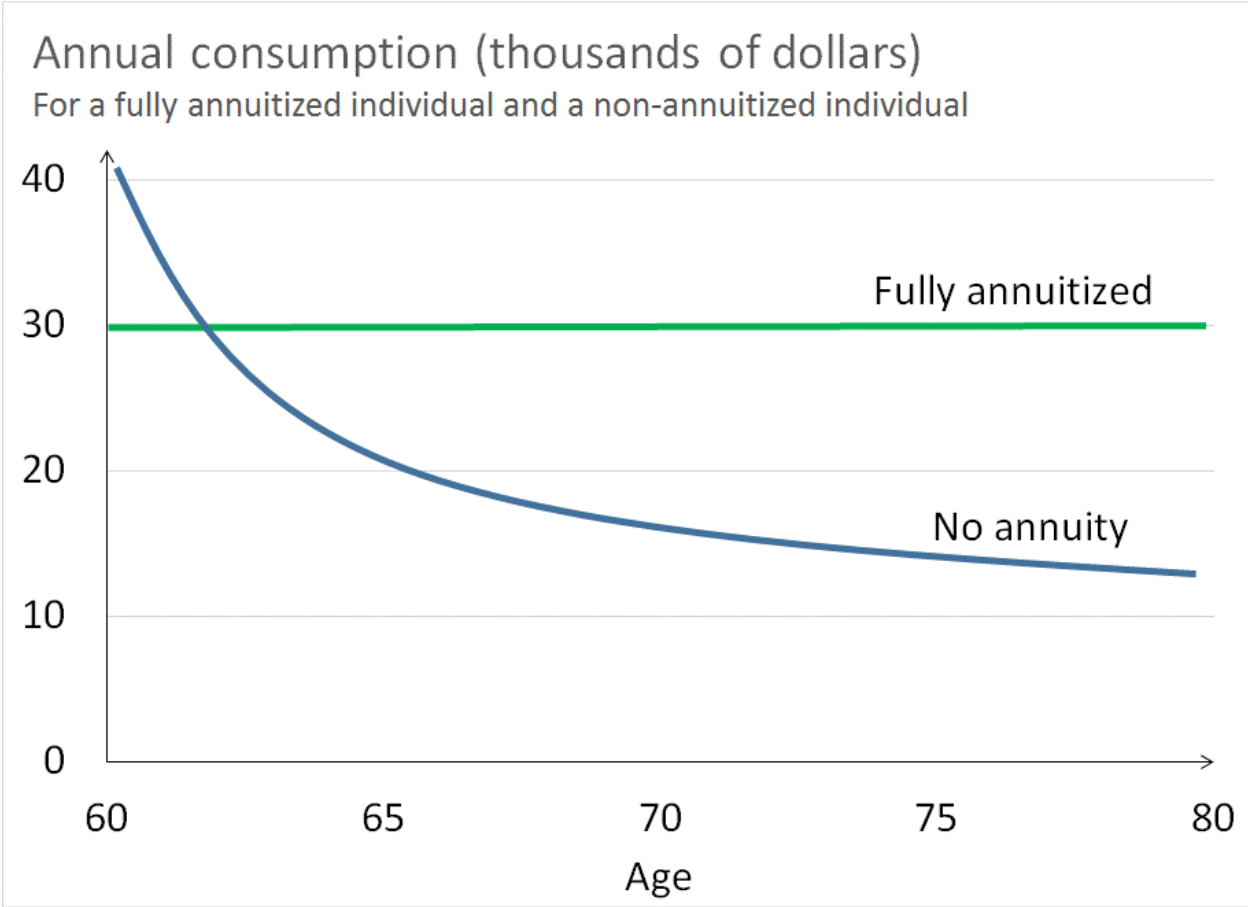
Notes: Columns (1) and (2) report the number of impaired activities of daily living (ADL) and the number of chronic conditions corresponding to each health state. Column (3)-(6) reports the life expectancy and quality of life (EQ-5D index) for an individual at ages 50 and 70. Columns (7)-(8) report the probability that an individual transitions to a different health state in the following year. Data source: Future Elderly Model.

Table 3. Value of treatment and prevention (in thousands of dollars) for a 50-year-old

Health state	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Life expectancy	Disc. life expectancy	VSL	VSI	WTP per discounted life-year		
					Treatment	Prevention	Treatment/Prevention
1	35.0	20.9	\$3,524	N/A	\$169	N/A	N/A
2	31.5	19.5	\$3,650	\$201	\$187	\$147	1.28
3	27.5	17.8	\$3,817	\$458	\$215	\$148	1.45
4	23.3	15.7	\$4,019	\$769	\$256	\$150	1.71
5	20.2	14.1	\$4,164	\$1,025	\$296	\$151	1.95
6	29.7	18.8	\$3,729	\$304	\$199	\$147	1.36
7	26.6	17.4	\$3,866	\$518	\$223	\$148	1.50
8	23.0	15.6	\$4,036	\$789	\$259	\$150	1.73
9	20.2	14.1	\$4,185	\$1,026	\$297	\$151	1.97
10	17.3	12.4	\$4,354	\$1,286	\$351	\$153	2.30
11	27.5	17.8	\$3,813	\$458	\$215	\$148	1.45
12	24.1	16.1	\$3,973	\$706	\$246	\$150	1.65
13	21.7	14.9	\$4,099	\$897	\$275	\$151	1.83
14	19.0	13.5	\$4,259	\$1,118	\$316	\$152	2.09
15	16.8	12.2	\$4,375	\$1,329	\$360	\$153	2.35
16	26.6	17.3	\$3,817	\$536	\$221	\$150	1.47
17	22.5	15.3	\$4,025	\$843	\$264	\$151	1.75
18	19.5	13.7	\$4,192	\$1,093	\$307	\$152	2.02
19	16.9	12.2	\$4,345	\$1,327	\$357	\$153	2.33
20	15.2	11.1	\$4,446	\$1,496	\$400	\$154	2.60

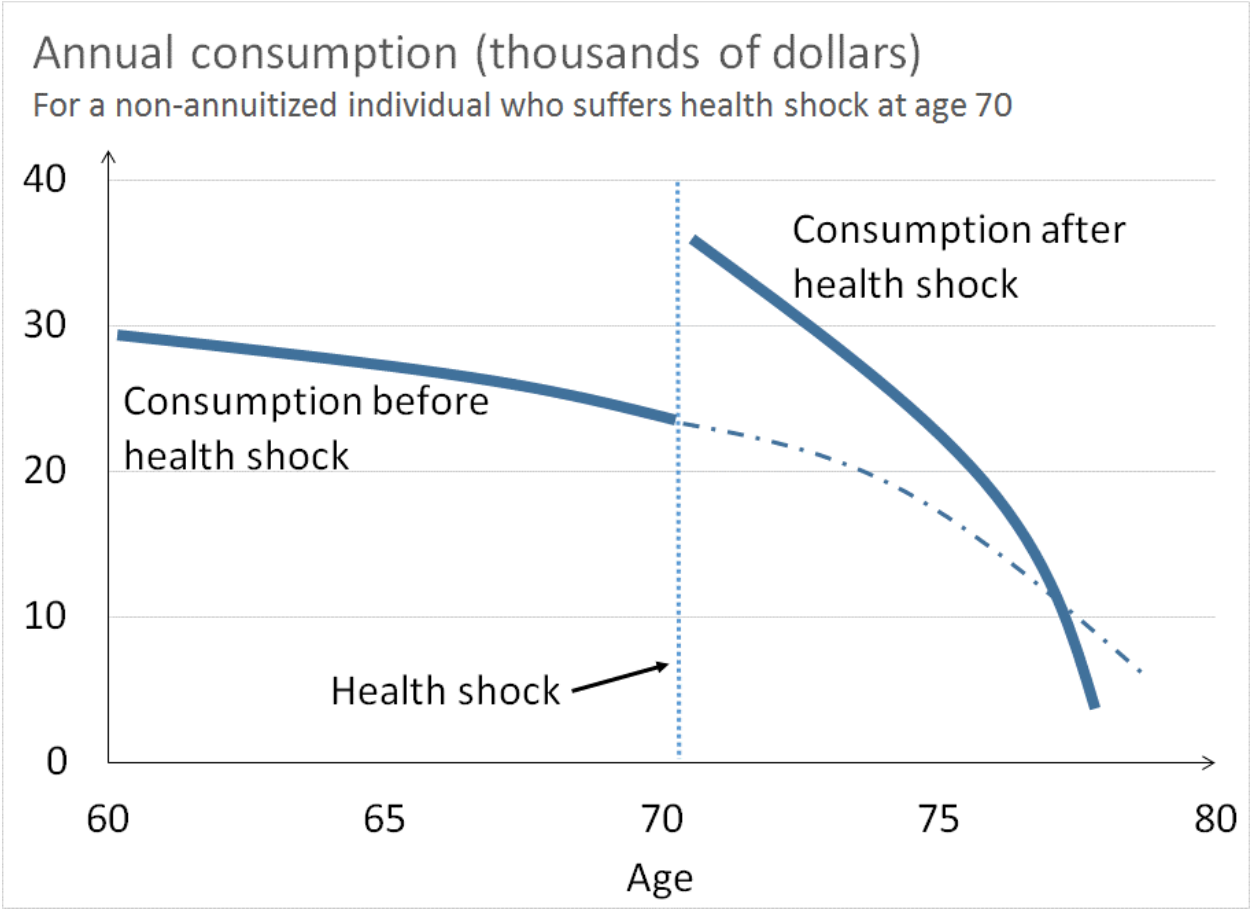
Notes: Table displays values (in thousands of dollars) from a life-cycle modeling exercise where mortality is stochastic. Column (1) reports remaining life expectancy for a 50-year-old for twenty different possible health states. Column (2) reports remaining life expectancy discounted at 3 percent. Column (3) reports value of statistical life (VSL) for an individual in each health state. Column (4) reports value of statistical illness (VSI) for a healthy individual in state 1. Column (5) reports a sick individual's willingness-to-pay (WTP) per discounted life-year for a therapeutic treatment, which is equal to the value in column (3) divided by the value in column (2). Column (6) reports the healthy individual's corresponding WTP for a preventive treatment, which is equal to the value in column (4) divided by the difference between 20.9 and the value in column (2). Column (7) reports the ratio of the values reported in columns (5) and (6).

Figure 1. Illustrative example: annual consumption for fully annuitized and non-annuitized consumers



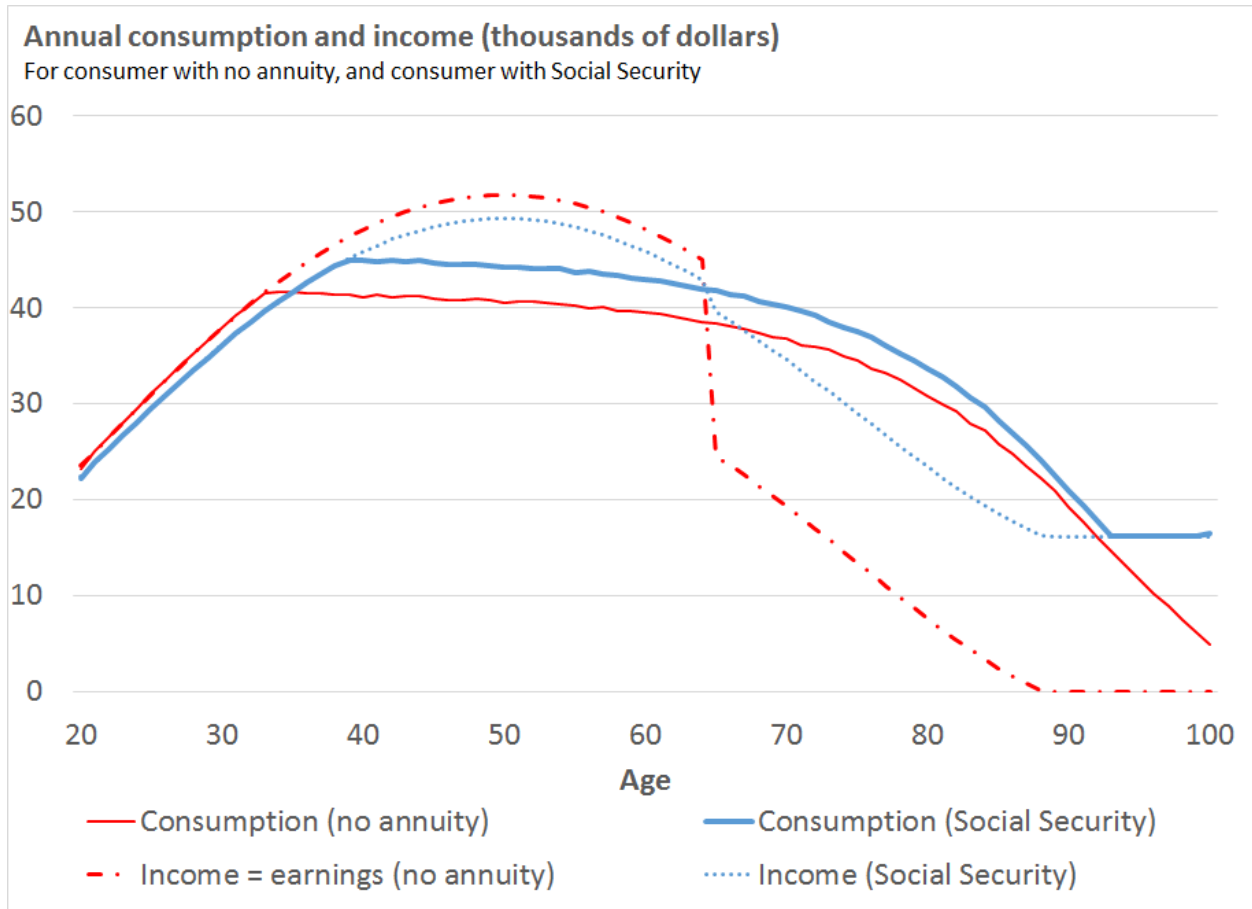
Notes: It is optimal for a non-annuitized consumer who is exposed to mortality risk to shift her consumption forward in time.

Figure 2. Illustrative example: health shocks induce a consumer to increase consumption when she is not annuitized



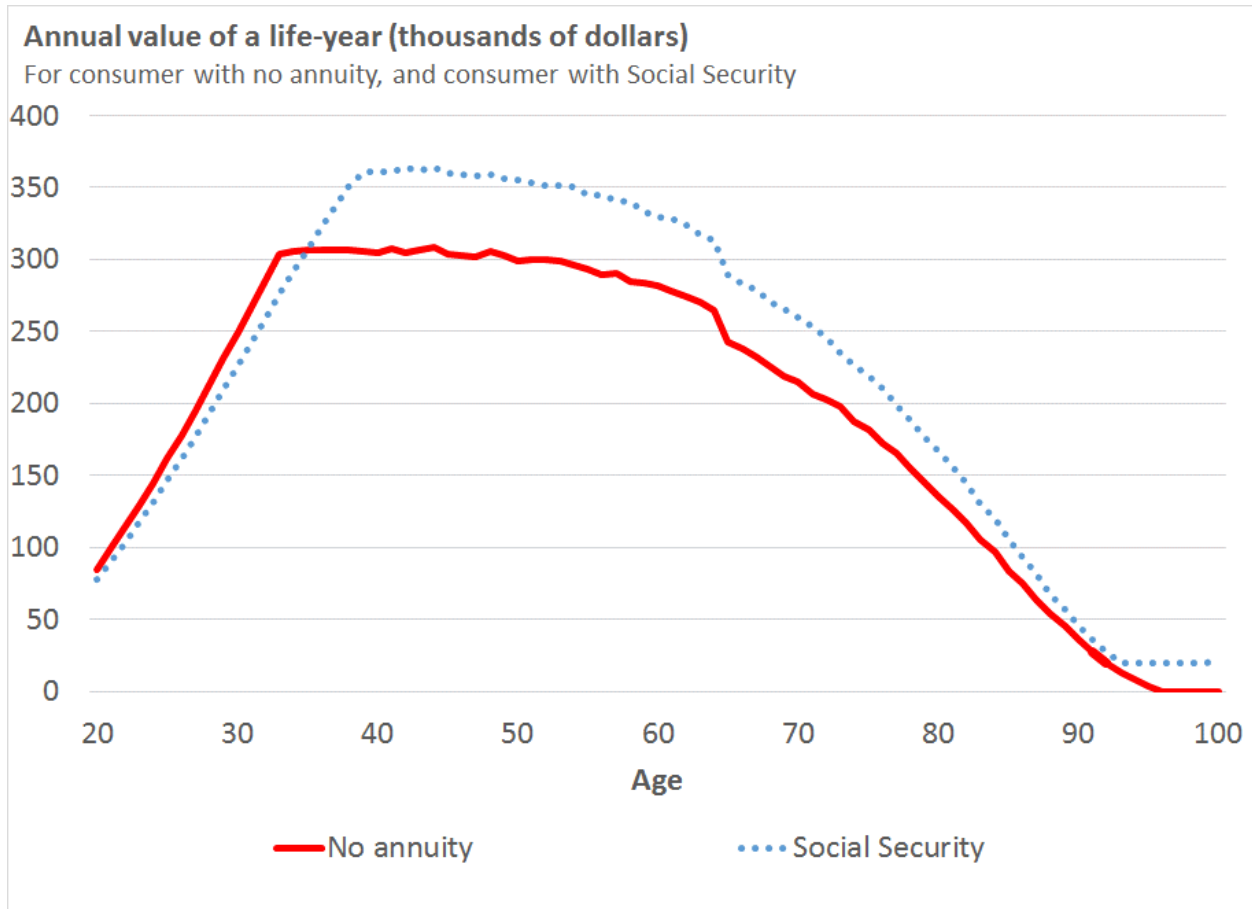
Notes: Consider a hypothetical, non-annuitized individual who suffers a health shock at age 70 that significantly increases her annual mortality. It is optimal for this consumer to “spend down” her wealth by increasing consumption.

Figure 3. Life-cycle profiles of consumption and income when mortality is deterministic



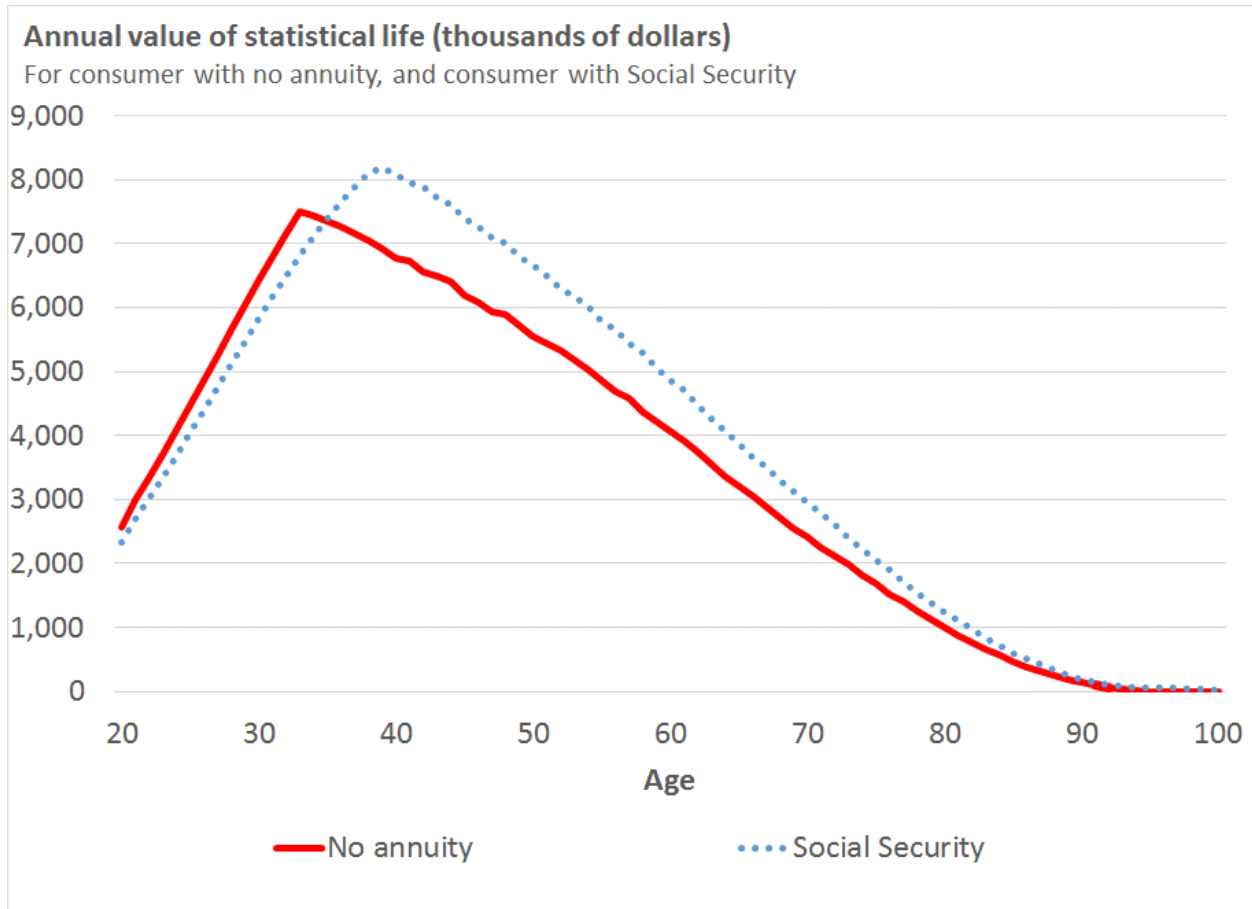
Notes: Figure plots consumption results from a life-cycle modeling exercise where mortality is deterministic. “Consumption (full annuity)” displays consumption (= income) when the consumer fully annuitizes all future earnings at age 20. “Consumption (Social Security)” displays consumption for a consumer receiving typical Social Security benefits that are financed by an earnings tax. “Consumption (no annuity)” displays consumption for a consumer whose income equals her earnings. The net present value at age 20 of all future income is the same across all three scenarios.

Figure 4. Life-cycle profile of the value of a life-year when mortality is deterministic



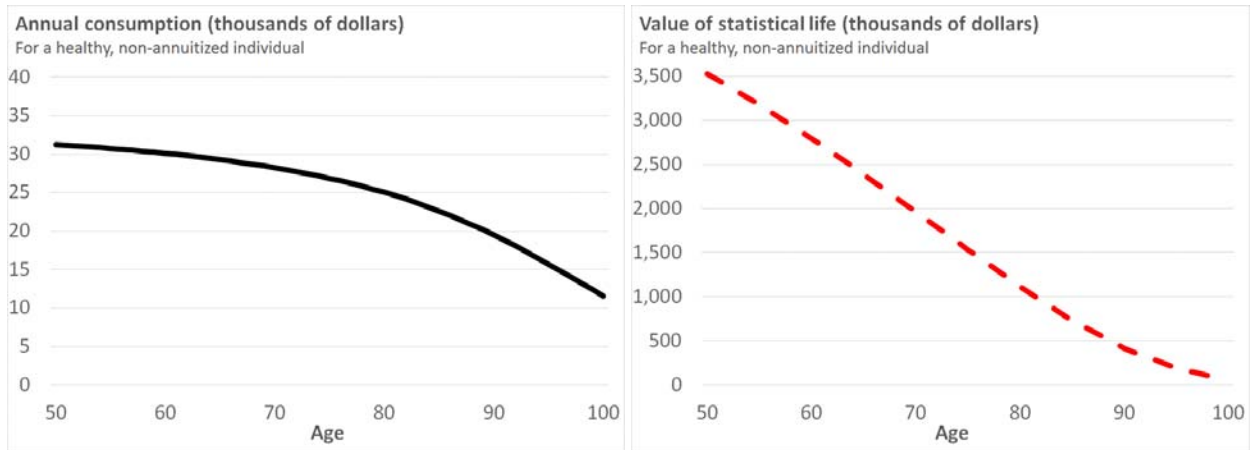
Notes: Figure plots the value of a life-year for the three scenarios displayed in Figure 3. “No annuity” assumes the consumer’s income equals her earnings. “Social Security” assumes the consumer receives typical Social Security benefits that are financed by an earnings tax. The net present value at age 20 of all future income is identical in all three scenarios.

Figure 5. Life-cycle profile of the value of statistical life when mortality is deterministic



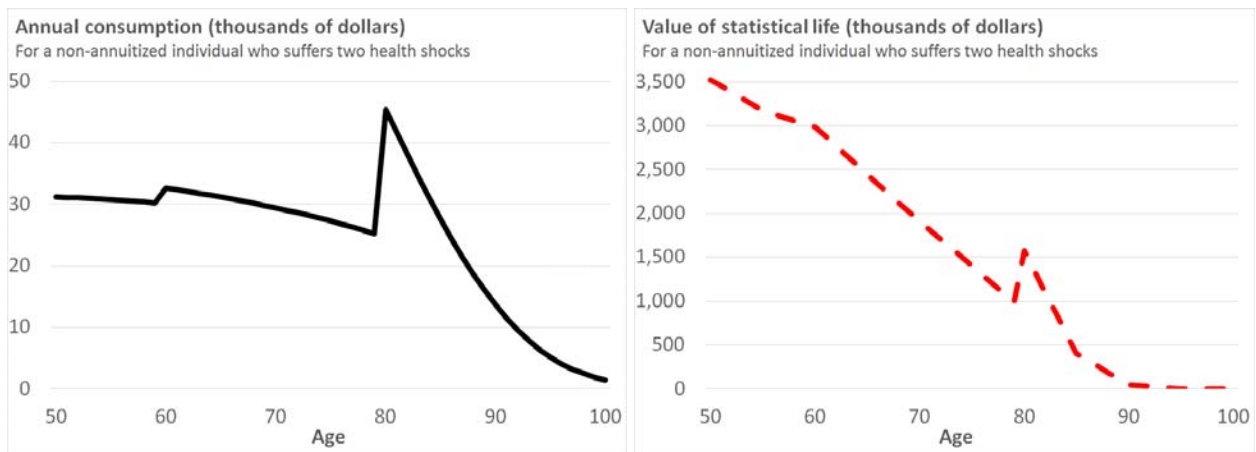
Notes: Figure plots the value of statistical life for the three scenarios displayed in Figure 3. “No annuity” assumes the consumer’s income equals her earnings. “Social Security” assumes the consumer receives typical Social Security benefits that are financed by an earnings tax. The net present value at age 20 of all future income is identical in all three scenarios.

Figure 6. Consumption and value of life when mortality is stochastic and individual is always healthy



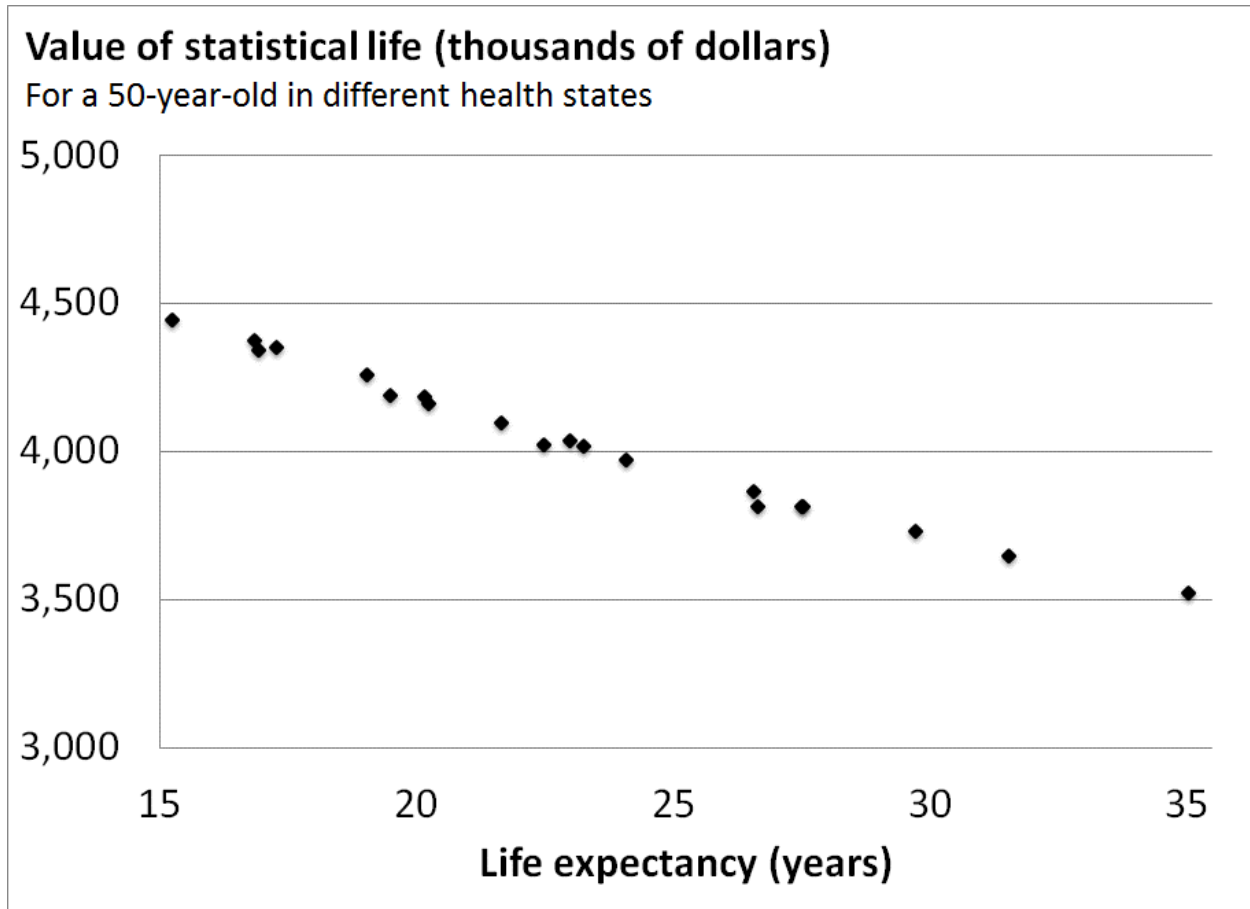
Notes: The left figure plots an individual’s consumption profile from a life-cycle modeling exercise where mortality is stochastic and the consumer never falls ill. The right figure plots the corresponding value of statistical life for this individual.

Figure 7. Consumption and value of life when mortality is stochastic and individual falls ill



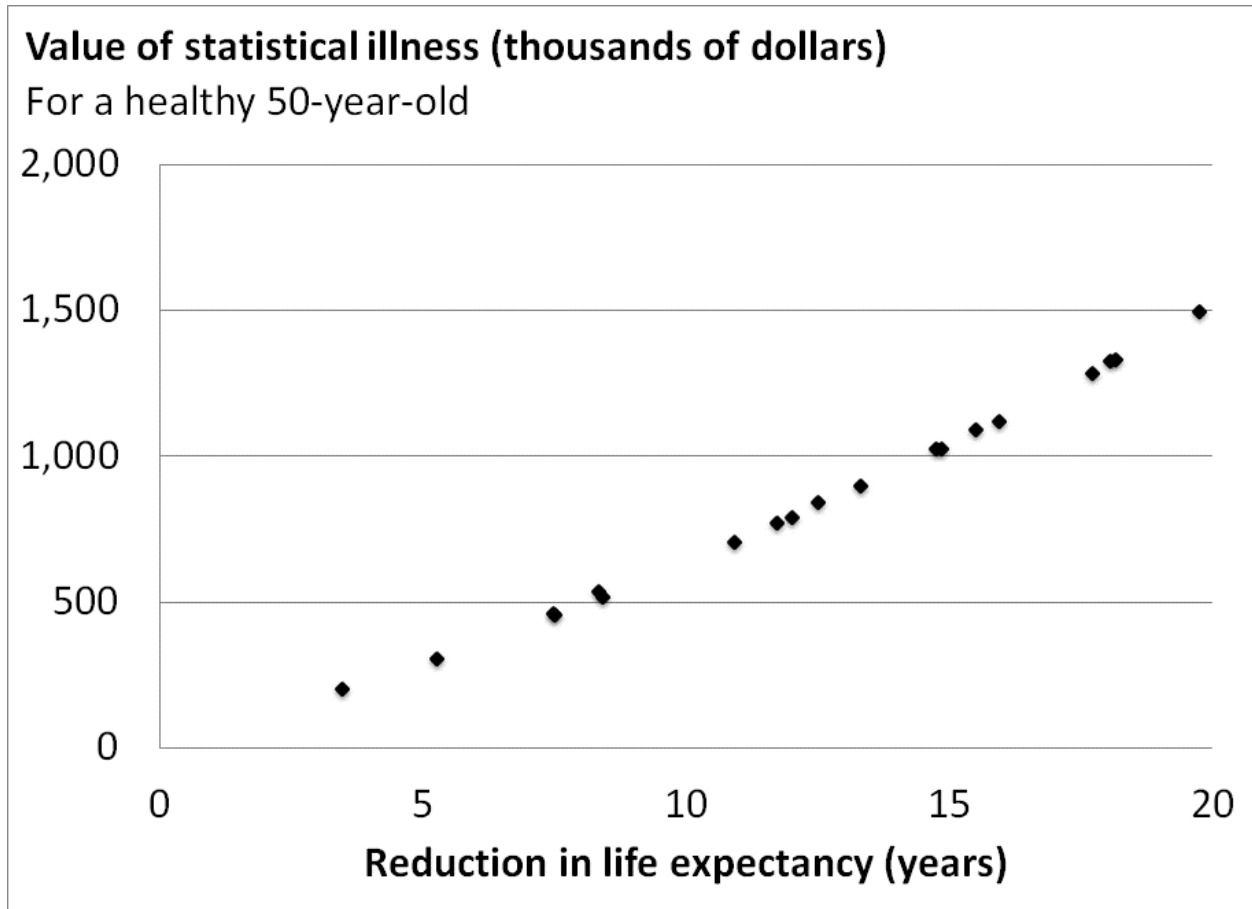
Notes: The left figure plots annual consumption for an individual who falls ill twice in her life. At age 60, this causes her difficulties with one routine activity of daily living (ADL). At age 80, she is diagnosed with two chronic conditions and subsequently has difficulties with two ADL’s. The right figure plots the corresponding value of statistical life (VSL) for this individual. The second illness is severe enough that it causes a 50 percent increase in her VSL.

Figure 8. Value of statistical life (VSL) at age 50, as a function of remaining life expectancy



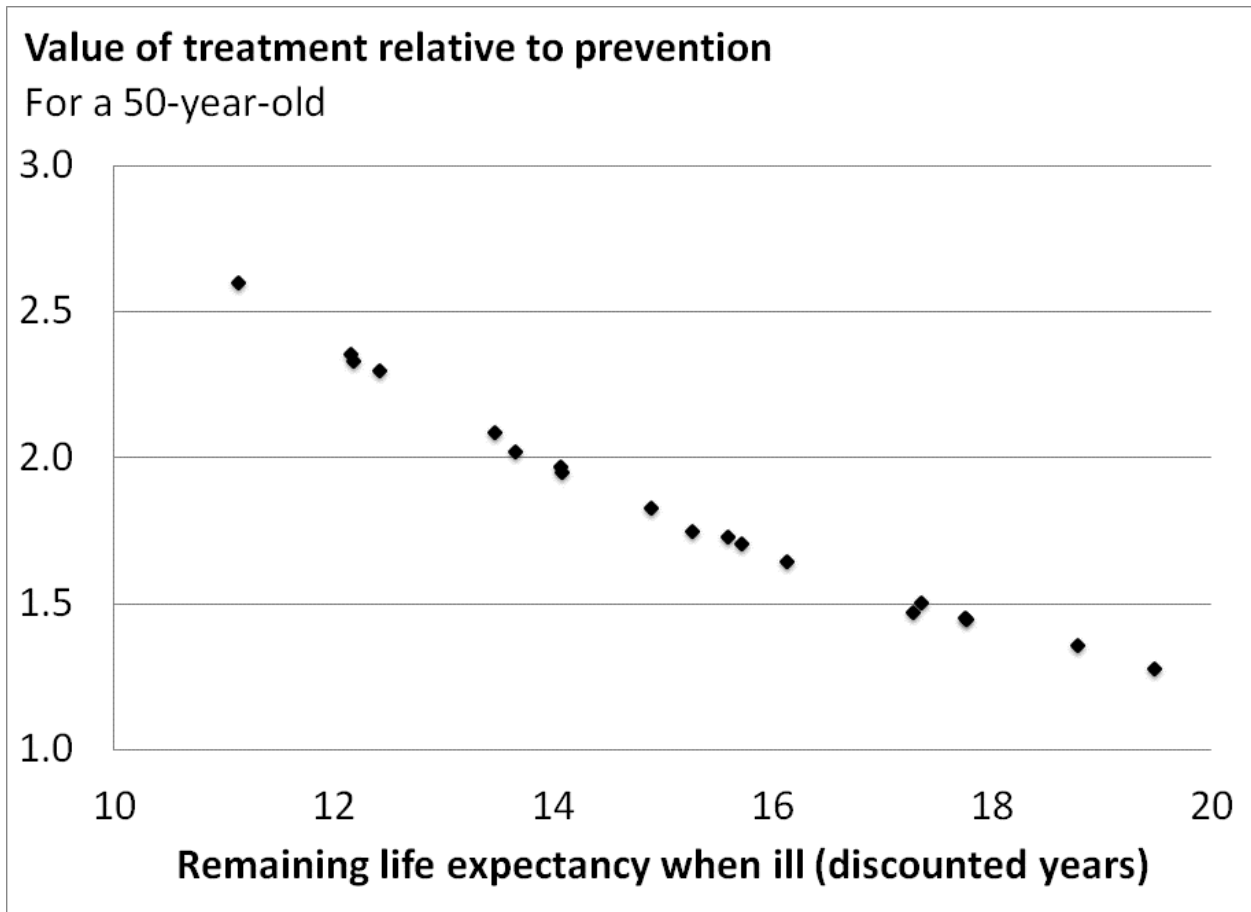
Notes: Figure plots VSL at age 50 for the twenty different health states employed in the stochastic mortality model. The remaining life expectancy in these states ranges from 15.2 years to 35.0 years. VSL ranges from \$3.5 million to \$4.4 million. These data are also reported in columns (1) and (3) Table 3.

Figure 9. Value of statistical illness (VSI) at age 50, as a function of illness severity



Notes: Figure plots VSI at age 50 for a healthy individual with a remaining life expectancy of 35 years who faces a risk of transitioning to one of 19 alternative, sicker health states. Each point represents the healthy individual's willingness-to-pay for a marginal reduction in the probability of transitioning to a particular sick state. These data are also reported in columns (1) and (4) of Table 3.

Figure 10. Value of treatment relative to prevention at age 50, as a function of (discounted) life expectancy in sick state



Notes: Figure plots the ratio of an individual's willingness-to-pay (WTP) per life-year for preventive care when healthy to the WTP per life-year for a treatment when sick, as a function of life expectancy when sick. Life expectancy is calculated using a discount rate of 3 percent. These data are also reported in columns (2) and (7) of Table 3.

APPENDIX (FOR ONLINE PUBLICATION ONLY)

Appendix A provides proofs for lemmas and propositions stated in the main text. Appendix B provides supporting details for the data and numerical calculations reported in Section IV of the main text. Appendix C provides derivations for the value of statistical life and the value of statistical illness for a fully annuitized consumer when mortality is stochastic.

A. Mathematical proofs of results from main text

Proof of Lemma 1:

Let $V(t, W(t), j)$ be taken as given (exogenous). Consider the deterministic optimization problem:

$$V(0, W_0, i) = \max_{c_i(t)} \left\{ \int_0^T e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) dt \right\}$$

subject to

$$\frac{\partial W(t)}{\partial t} = rW(t) + m_i(t) - c_i(t)$$

Denote the optimal value-to-go as

$$\tilde{V}(u, W(u), i) = \max_{c_i(t)} \left\{ \int_u^T e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) dt \right\}$$

Setting $\tilde{V}(u, W(u), i) = e^{-\rho t} \tilde{S}(i, t) V(t, W(t), i)$ then demonstrates that $V(\cdot)$ satisfies the HJB (12) for i . See Parpas and Webster (2013) for additional details.

QED

Proof of Lemma 3:

The proof proceeds by induction on $i \leq n$. For the base case $i = n$, in which no state transitions are possible, the solution to the costate equation (given in the main text) simplifies to:

$$\begin{aligned} p_\tau^{(n)} &= \theta^{(n)} e^{-r\tau} = \exp \left\{ - \int_0^\tau \rho + \bar{\mu}_n(s) ds \right\} u_c(c_n(\tau), q_n(\tau)) \\ &= \theta^{(n)} e^{-r\tau} e^{-r(\tau-t)} \\ &= p_t^{(n)} e^{-r(\tau-t)} \\ &= \exp \left\{ - \int_0^t \rho + \bar{\mu}_n(s) ds \right\} u_c(c_n(t), q_n(t)) e^{-r(\tau-t)} \end{aligned}$$

This then implies that

$$u_c(c_n(t), q_n(t)) = e^{r(\tau-t)} e^{-\rho(\tau-t)} \exp \left\{ - \int_t^\tau \bar{\mu}_n(s) ds \right\} u_c(c_n(\tau), q_n(\tau))$$

which shows that the lemma holds for $i = n$.

For the induction step, suppose the lemma is true for $j > i$, $1 \leq i \leq n - 1$. For any subinterval $[0, \tau]$, the solution of the costate equation can be written as:

$$p_t^{(i)} = \left[\int_t^\tau e^{(r-\rho)s} \exp \left\{ - \int_0^s \bar{\mu}_i(u) + \sum_{j \neq i} \lambda_{ij}(u) du \right\} \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds \right] e^{-rt} + \theta(\tau, i) e^{-r\tau} \quad (\text{A1})$$

where $\theta(\tau, i)$ is a constant that depends on the choice of τ and i . (Take the derivative of $p_t^{(i)}$ with respect to t to verify.) Evaluating equation (A1) at $t = \tau$ and combining with equation (14) from the main text yields:

$$p_\tau^{(i)} = \theta(\tau, i) e^{-r\tau} = \exp \left\{ - \int_0^\tau \rho + \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} u_c(c_i(\tau), q_i(\tau))$$

which implies

$$\theta(\tau, i) = e^{(r-\rho)\tau} \exp \left\{ - \int_0^\tau \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} u_c(c_i(\tau), q_i(\tau)) \quad (\text{A2})$$

Also, from equation (14) we know that:

$$p_t^{(i)} = \exp \left\{ - \int_0^t \rho + \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} u_c(c_i(t), q_i(t))$$

Plugging equations (14) and (A2) into equation (A1) yields:

$$\begin{aligned} & u_c(c_i(t), q_i(t)) \exp \left\{ - \int_0^t \rho + \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} \\ &= \left[\int_t^\tau e^{(r-\rho)s} \exp \left\{ - \int_0^s \bar{\mu}_i(u) + \sum_{j \neq i} \lambda_{ij}(u) du \right\} \sum_{j \neq i} \lambda_{ij}(s) \frac{\partial V(s, W(s), j)}{\partial W(s)} ds \right] e^{-rt} \\ &+ e^{-rt} e^{(r-\rho)\tau} \exp \left\{ - \int_0^\tau \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} u_c(c_i(\tau), q_i(\tau)) \end{aligned}$$

Since $\frac{\partial V(s, W(s), j)}{\partial W(s)} = u_c(c_j(s), q_j(s))$, we obtain:

$$\begin{aligned} u_c(c_i(t), q_i(t)) &= \int_t^\tau e^{(r-\rho)(s-t)} \exp \left\{ - \int_t^s \bar{\mu}_i(u) + \sum_{j \neq i} \lambda_{ij}(u) du \right\} \sum_{j \neq i} \lambda_{ij}(s) u_c(c_j(s), q_j(s)) ds \\ &+ e^{(r-\rho)(\tau-t)} \exp \left\{ - \int_t^\tau \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} u_c(c_i(\tau), q_i(\tau)) \end{aligned}$$

$$\begin{aligned}
&= \int_t^\tau e^{(r-\rho)(s-t)} \exp \left\{ - \int_t^s \bar{\mu}_i(u) + \sum_{j \neq i} \lambda_{ij}(u) du \right\} \sum_{j \neq i} \lambda_{ij}(s) \mathbb{E} \left[e^{(r-\rho)(\tau-s)} \exp \left\{ - \int_s^\tau \mu(s) ds \right\} u_c(c_{y_\tau}(\tau), q_{y_\tau}(\tau)) \middle| Y_s = j \right] ds \\
&\quad + e^{(r-\rho)(\tau-t)} \exp \left\{ - \int_t^\tau \bar{\mu}_i(s) + \sum_{j \neq i} \lambda_{ij}(s) ds \right\} u_c(c_i(\tau), q_i(\tau)) \\
&= \mathbb{E} \left[e^{(r-\rho)(\tau-s)} \exp \left\{ - \int_t^\tau \mu(s) ds \right\} u_c(c_{y_\tau}(\tau), q_{y_\tau}(\tau)) \middle| Y_t = i \right]
\end{aligned}$$

where the second equality follows from the induction hypothesis.

QED

Proof of Proposition 4:

Choosing once again the Dirac delta function for $\delta(\cdot)$ in **Lemma 2** yields

$$\begin{aligned}
\frac{\partial \mathbb{E}U}{\partial \varepsilon} \Big|_{\varepsilon=0} &= \int_0^T \left[e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, W(t), j) \right) \right] dt \\
&= \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) u(c_{y_t}(t), q_{y_t}(t)) dt \middle| Y_0 = i \right]
\end{aligned}$$

Dividing the result by the marginal utility of wealth at time $t = 0$ then yields the value of statistical life given by equation (15):

$$VSL = \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) \frac{u(c_{y_t}(t), q_{y_t}(t))}{u(c_{Y_0}(0), q_{Y_0}(0))} dt \middle| Y_0 = i \right] = \int_0^T e^{-rt} v(i, t) dt$$

Applying **Lemma 3** for $t = 0$ allows us to rewrite VSL as

$$\begin{aligned}
VSL(i) &= \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) \frac{u(c_{y_t}(t), q_{y_t}(t))}{\mathbb{E} \left[e^{(r-\rho)t} \exp \left\{ - \int_0^t \mu(s) ds \right\} u_c(c_{Y_t}(t), q_{Y_t}(t)) \middle| Y_0 \right]} dt \middle| Y_0 = i \right] \\
&= \mathbb{E} \left[\int_0^T e^{-rt} \frac{S(t) u(c_{y_t}(t), q_{y_t}(t))}{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} u_c(c_{Y_t}(t), q_{Y_t}(t)) \middle| Y_0 \right]} dt \middle| Y_0 = i \right]
\end{aligned}$$

which by exchanging expectation and integration shows that the value of a life-year, $v(i, t)$, is equal to the expected utility of consumption normalized by the expected marginal utility of consumption:

$$v(i, t) = \frac{\mathbb{E} \left[S(t) u(c_{y_t}(t), q_{y_t}(t)) \middle| Y_0 = i \right]}{\mathbb{E} \left[S(t) u_c(c_{y_t}(t), q_{y_t}(t)) \middle| Y_0 = i \right]}$$

QED

Proof of Proposition 5:

The proposition assumes there are $n = 2$ states, with $\bar{\mu}_2(s) > \bar{\mu}_1(s) \forall s$. That is, health in state 2 is strictly worse than health in state 1. For simplicity, we abstract from quality of life, $q(t)$. Without loss of generality, we will prove the proposition for the case where the consumer transitions from state 1 to state 2 at time $t = 0$.

For state 2, the solution to the costate equation is:

$$p_t^{(2)} = \theta^{(2)} e^{-rt}$$

and from the first-order condition (14) we obtain:

$$p_t^{(2)} = e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_2(s) ds \right\} u_c(c_2(t))$$

The two preceding equations imply that

$$u_c(c_2(t)) = \theta^{(2)} e^{(\rho-r)t} \exp \left\{ \int_0^t \bar{\mu}_2(s) ds \right\}$$

For state 1, the costate equation is:

$$\begin{aligned} \dot{p}_t^{(1)} &= -p_t^{(1)} r - e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_1(s) + \lambda_{12}(s) ds \right\} \lambda_{12}(t) \frac{\partial V(t, W(t), 2)}{\partial W(t)} \\ &= -p_t^{(1)} r - e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_1(s) + \lambda_{12}(s) ds \right\} \lambda_{12}(t) u_c(c_2(t)) \\ &= -p_t^{(1)} r - e^{-rt} \exp \left\{ - \int_0^t \lambda_{12}(s) ds \right\} \lambda_{12}(t) \theta^{(2)} \exp \left\{ \int_0^t \bar{\mu}_2(s) - \bar{\mu}_1(s) ds \right\} \end{aligned} \quad (\text{A3})$$

Before proceeding, we first prove the following two lemmas.

Appendix Lemma A1:

There exists a $t \in [0, T]$ such that

$$p_t^{(1)} \geq \exp \left\{ - \int_0^t \lambda_{12}(s) ds \right\} p_t^{(2)}$$

Proof of Appendix Lemma A1:

Suppose by way of contradiction that $p_t^{(1)} < \exp \left\{ - \int_0^t \lambda_{12}(s) ds \right\} p_t^{(2)} \forall t \in [0, T]$. Then, since $\bar{\mu}_2(s) > \bar{\mu}_1(s)$ we have

$$e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_2(s) ds \right\} p_t^{(1)} < e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_1(s) ds \right\} \exp \left\{ - \int_0^t \lambda_{12}(s) ds \right\} p_t^{(2)}$$

Rearranging then yields

$$u_c(c_1(t)) = \frac{p_t^{(1)}}{e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_1(s) ds \right\} \exp \left\{ - \int_0^t \lambda_{12}(s) ds \right\}} < \frac{p_t^{(2)}}{e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_2(s) ds \right\}} = u_c(c_2(t))$$

which implies $c_2(t) < c_1(t) \forall t$. But then we have a contradiction: $c_2(t)$ cannot be an optimal consumption plan because the feasible consumption plan $c_1(t)$ strictly dominates $c_2(t)$.

QED

Appendix Lemma A2:

$$p_0^{(1)} > \theta^{(2)} = p_0^{(2)}$$

Proof of Appendix Lemma A2:

Define

$$g(t) = \exp\left\{-\int_0^t r + \lambda_{12}(s) ds\right\} \theta^{(2)} = \exp\left\{-\int_0^t \lambda_{12}(s) ds\right\} p_t^{(2)}$$

Differentiating with respect to t yields

$$\begin{aligned} \dot{g}(t) &= -g(t)r - \exp\left\{-rt - \int_0^t \lambda_{12}(s) ds\right\} \lambda_{12}(t) \theta^{(2)} \\ &= \phi(g(t), t) \end{aligned}$$

Combining this result with equation (A3) then yields the following inequality:

$$\dot{p}_t^{(1)} < \phi(p_t^{(1)}, t)$$

Suppose by way of contradiction that $p_0^{(1)} < \theta^{(2)} = g(0)$. Then by standard comparison arguments for ordinary differential equations, we have $p_t^{(1)} < g(t) = \exp\left\{-\int_0^t \lambda_{12}(s) ds\right\} p_t^{(2)} \forall t \in [0, T]$, which is a contradiction to the result from **Appendix Lemma A1**.

QED

Thus, we have

$$u_c(c_1(0)) = p_0^{(1)} > p_0^{(2)} = u_c(c_2(0))$$

which implies

$$c_2(0) > c_1(0)$$

QED

Proof of Proposition 6:

Without loss of generality, consider the case $t = 0$. From **Proposition 5** and **Appendix Lemmas A1 and A2**, it is clear that $c_2(0) > c_1(0)$, $c_2(t) \geq c_1(t)$ for $t \leq t_0$, and $c_2(t) \leq c_1(t)$ for $t > t_0$. Making use of the assumption that no state transitions occur for $t > 0$, we have that

$$\begin{aligned} VSL(2,0) &= \int_0^T e^{-rt} \frac{S_2(t)u(c_2(t))}{S_2(t)u_c(c_2(t))} dt \\ &= \int_0^T e^{-rt} \frac{u(c_2(t))}{u_c(c_2(t))} dt \end{aligned}$$

and

$$VSL(1,0) = \int_0^T e^{-rt} \frac{u(c_1(t))}{u_c(c_1(t))} dt$$

Let $Y(x) = \frac{u(x)}{u_c(x)}$. Under the stated assumptions, we have that

$$Y'(x) = 1 - \frac{u(x)u_{cc}(x)}{(u_c(x))^2} > 0$$

$$Y''(x) = \frac{2(u_{cc}(x))^2 u(x) - u_c^2(x)u_{cc}(x) - u_c(x)u(x)u_{ccc}(x)}{(u_c(x))^3} > 0$$

Employing Taylor's theorem then yields:

$$\begin{aligned}
VSL(2,0) &= \int_0^T e^{-rt} Y(c_2(t)) dt \\
&= \int_0^T e^{-rt} \left[Y(c_1(t)) + [c_2(t) - c_1(t)]Y'(c_1(t)) + \underbrace{\frac{1}{2}[c_2(t) - c_1(t)]^2 Y''(c_1(t))}_{>0} \right] dt \\
&> \int_0^T e^{-rt} Y(c_1(t)) dt + \int_0^{t_0} e^{-rt} Y'(c_1(t)) \underbrace{[c_2(t) - c_1(t)]}_{\geq 0} dt \\
&\quad + \int_{t_0}^T e^{-rt} Y'(c_1(t)) \underbrace{[c_2(t) - c_1(t)]}_{\leq 0} dt \\
&> \int_0^T e^{-rt} Y(c_1(t)) dt + \int_0^{t_0} e^{-rt} Y'(c_1(t_0)) [c_2(t) - c_1(t)] dt \\
&\quad + \int_0^{t_0} e^{-rt} Y'(c_1(t_0)) [c_2(t) - c_1(t)] dt \\
&= \int_0^T e^{-rt} Y(c_1(t)) dt + Y'(c_1(t_0)) \underbrace{\left[\int_0^T e^{-rt} c_2(t) dt - \int_0^T e^{-rt} c_1(t) dt \right]}_{=0} \\
&= \int_0^T e^{-rt} Y(c_1(t)) dt \\
&= VSL(1,0)
\end{aligned}$$

where the final step follows from the budget constraint.

QED

B. Data and derivations for numerical models

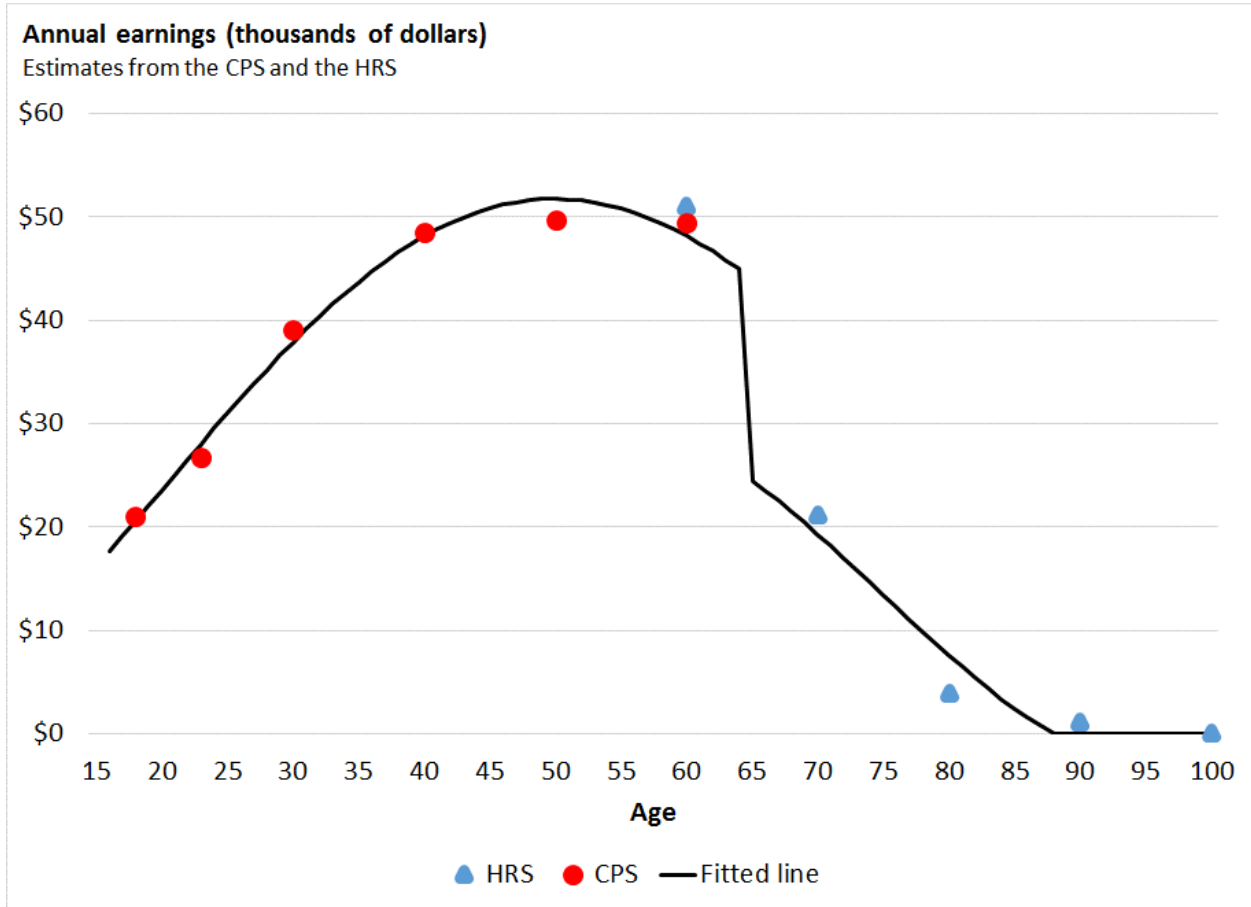
B1. Earnings data

We obtain earnings data for employed individuals under the age of 65 from the 2016 Current Population Survey (CPS).¹⁷ We also obtain earnings data for respondents over the age of 55 from the 2014 Health and Retirement Survey (HRS). For both surveys, the data represent earnings before taxes and other deductions, and include wages, salaries, and tips. The HRS earnings data also include self-employment income. (The CPS data exclude self-employed individuals.)

The CPS earnings data are binned into the following age groups: 16-19, 20-24, 25-34, 35-44, 45-54, and 55-64. We collapse the HRS earnings data into the following age groups: 55-64, 65-74, 75-84, 85-94, and 95-104. The resulting estimates are plotted in Appendix Figure 1. We smooth the data by fitting it to a quartic polynomial, and include an indicator variable for ages over 65. The dependent variable in the regression is the CPS earnings estimate for ages under 65, and the HRS estimate for ages over 65. Finally, we constrain the fitted prediction to be non-negative.

¹⁷ These data are available at <http://data.bls.gov/pdq/querytool.jsp?survey=le>.

Appendix Figure 1. Annual earnings estimates from CPS and HRS



Notes: Figure plots annual earnings by midpoint of age group as estimated by the 2016 Current Population Survey (CPS) for respondents under age 65 and the 2014 Health and Retirement Survey (HRS) for respondents over age 55. The fitted line corresponds to a regression of annual earnings on a quartic polynomial in age and an indicator equal to 1 for ages 65 and over. The dependent variable, annual earnings, corresponds to CPS estimates for ages under 65 and HRS estimates for ages over 65.

B2. Deterministic mortality

We employ dynamic programming techniques to solve for the optimal consumption path. The value function is defined as:

$$V(t, w_t) = \max_{\{c_t\}} \sum_{s=t}^T e^{-\rho(s-t)} S_t(s) u(c_s)$$

We can use the value function to rewrite the optimization problem as a recursive Bellman equation:

$$V(t, w_t) = \max_{\{c_t\}} u(c_t) + \frac{1 - d_t}{e^\rho} V(t + 1, w_{t+1})$$

Once we have solved for the optimal consumption path, we can use the analytical formulas derived in the main text to calculate the value of life.

Our calibration exercises use numerical methods to solve the following Bellman equation:

$$V(t, w_t) = \max_{\{c_t\}} u(c_t) + \frac{1 - d_t}{e^\rho} V(t + 1, w_{t+1})$$

Because the problem is finite, we can work backwards from the final period. We discretize the state space into $N_w = 2,000$ points evenly distributed across the interval $[0, w_{max}]$. Let that set of values be $\{w_n\}$. Define $g_t(w_t) = w_{t+1}$ as a mapping from the current wealth state, w_t , to the optimal wealth state in the following period, w_{t+1} .

It is clear that the consumer should consume all her wealth in the final period, i.e., $g_T(w_T) = 0$ for all $w_T \in \{w_n\}$. This implies that $V(T, w_T) = u(w_T + y_T)$ for all $w_T \in \{w_n\}$.

Next, we calculate $V(T - 1, w_{T-1}) = \max_{g(w_{T-1})=w_T} u(w_{T-1} + y_{T-1} - w_T/e^r) + \frac{1-d_{t+1}}{e^\rho} V(T, w_T)$. In other words, for each $w_{T-1} \in \{w_n\}$, we calculate the optimal $V(T - 1, w_{T-1})$ by determining which choice of $g_{T-1}(w_{T-1}) = w_T \in \{w_n\}$ will maximize utility. This algorithm is then repeated for $t = T - 2, T - 3, \dots, 1$.

Given the initial condition, w_1 , we can then employ our results to calculate $w_2 = g_1(w_1)$, $w_3 = g_2(w_2), \dots, w_T$. Period consumption, c_t , is then calculated using the equation for the budget constraint.

B3. Stochastic mortality

We focus on the case where the consumer does not have access to annuities. We ignore income, and assume that all of consumer's wealth is available at time $t = 0$. This will allow us to generate an analytic solution to the consumer's problem, given by:

$$\max_{\{c_t\}} \mathbb{E}_0 \left[\sum_{t=0}^T e^{-\rho t} S_0(t) u(c_t, q_t) \right]$$

where

$$w_0 \text{ given,}$$

$$w_t \geq 0,$$

$$w_t = (w_{t-1} - c_{t-1})e^r$$

The utility function is

$$u(c, q) = q \frac{c^{1-\gamma}}{1-\gamma} - \frac{\underline{c}^{1-\gamma}}{1-\gamma}$$

where \underline{c} is the subsistence level of consumption for a healthy person. Because optimal consumption is unaffected by affine transformations of utility, we will assume $u(c, q) = qc^{1-\gamma}/(1-\gamma)$ when solving the model for consumption.

Define the value function

$$V(t, w_t, Y_t) = \max_{\{c_s\}} \mathbb{E} \left[\sum_{s=t}^T e^{-\rho(s-t)} S_t(s) u(c_s, q_{Y_s}(s)) \middle| Y_t, w_t, \text{alive} \right]$$

subject to

$$w_s = (w_{s-1} - c_{s-1})e^r, s > t, w_s \geq 0$$

Then we obtain the following Bellman equation:

$$V(t, w, i) = \max_{c_t} \left\{ u(c_t, q_i(t)) + e^{-\rho} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) V(t+1, (w - c_t)e^r, j) \right\}$$

where $V(T, w, i) = q_i(T)w^{1-\gamma}/(1-\gamma)$.

Appendix Proposition B1:

The value function and the optimal consumption level satisfy

$$V(t, w, i) = \frac{w^{1-\gamma}}{1-\gamma} K_{t,i}$$

$$c^*(t, w, i) = w \cdot c_{t,i}$$

where

$$c_{t,i} = \left[1 + \left(\frac{e^{-\rho+r(1-\gamma)} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j})}{q_i(t)} \right)^{\frac{1}{\gamma}} \right]^{-1}, t < T$$

$$c_{T,i} = 1$$

and $K_{t,i}$ satisfies the recursion:

$$K_{t,i} = \frac{q_i(t) e^{-\rho+r(1-\gamma)} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j})}{\left[(q_i(t))^{-1/\gamma} + \left[e^{(1-\gamma)r-\rho} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j}) \right]^{-1/\gamma} \right]^{-\gamma}}, t < T$$

$$K_{T,i} = q_i(T)$$

Proof of Appendix Proposition B1: see end of appendix B

When calculating VSL, we incorporate subsistence consumption back into the utility function. We first note that

$$\begin{aligned} V(0, w, i) &= \sum_{t=0}^T e^{-\rho t} \mathbb{E}_{0,i} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} \left(q_{Y_t}(t) \frac{c_{Y_t}^{1-\gamma}}{1-\gamma} - \frac{\underline{c}^{1-\gamma}}{1-\gamma} \right) \right] \\ &= \frac{1}{1-\gamma} \sum_{t=0}^T e^{-\rho t} \left\{ \mathbb{E}_{0,i} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_{Y_t}(t) c_{Y_t}^{1-\gamma} \right] - \underline{c}^{1-\gamma} \mathbb{E}_{0,i} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} \right] \right\} \end{aligned}$$

$$= \frac{1}{1-\gamma} \left[w^{1-\gamma} K_{0,i} - \underline{c}^{1-\gamma} \underbrace{\sum_{t=0}^T e^{-\rho t} \mathbb{E}_{0,i} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} \right]}_{\text{life expect. in state } i, \text{ discounted at rate } \rho} \right]$$

From the theory presented in the main text of the paper, we obtain:

$$\begin{aligned} VSL &= \mathbb{E} \left[\sum_{t=0}^T \exp \left\{ - \int_0^t \rho + \mu(s) ds \right\} \frac{u(c_{Y_t}(t), q_{Y_t}(t))}{u_c(c_{Y_0}(0), q_{Y_0}(0))} \middle| Y_0 \right] \\ &= \sum_{t=0}^T e^{-rt} \frac{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} u(c_{Y_t}(t), q_{Y_t}(t)) \middle| Y_0 \right]}{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} u_c(c_{Y_t}(t), q_{Y_t}(t)) \middle| Y_0 \right]} \\ &= \sum_{t=0}^T e^{-rt} \frac{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} \left(q_{Y_t}(t) \frac{c_{Y_t}^{1-\gamma}}{1-\gamma} - \frac{\underline{c}^{1-\gamma}}{1-\gamma} \right) \middle| Y_0 \right]}{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_{Y_t}(t) c_{Y_t}^{-\gamma} \middle| Y_0 \right]} \end{aligned}$$

or

$$VSL = \frac{1}{1-\gamma} \sum_{t=0}^T e^{-rt} \frac{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_{Y_t}(t) c_{Y_t}^{1-\gamma} \middle| Y_0 \right] - \underline{c}^{1-\gamma} \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} \middle| Y_0 \right]}{\underbrace{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_{Y_t}(t) c_{Y_t}^{-\gamma} \middle| Y_0 \right]}_{\bar{v}(t)}}$$

To evaluate this expression for VSL, we will make use of the following lemma.

Appendix Lemma B2: Let $W_{t,j}(\Psi) = \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} w_t^\Psi \mathbf{1}\{Y_t = j\} \middle| Y_0 \right]$ for $Y \in (1, \infty)$. Then $W_{t,j}(\Psi)$ satisfies the following recursion:

$$\begin{aligned} W_{0,Y_0}(\Psi) &= w_0^\Psi, W_{0,i}(\Psi) = 0, i \neq Y_0 \\ W_{t+1,j}(\Psi) &= e^{r\Psi} \sum_{k=1}^n W_{t,k}(\Psi) (1 - c_{t,k})^\Psi (1 - \bar{d}_k(t)) p_{k,j}(t) \end{aligned}$$

Proof of Appendix Lemma B2: see end of appendix B

Note that for $\Psi = 0$, the expression $\sum_{j=1}^n W_{t,j}(0) = \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} \middle| Y_0 \right]$ is simply the t -year survival probability. Using this **Appendix Lemma B2**, we obtain:

Appendix Proposition B3:

$$VSL_{Y_0} = \frac{1}{1-\gamma} \sum_{t=0}^T e^{-rt} \frac{\sum_{j=1}^n q_j(t) c_{t,j}^{1-\gamma} W_{t,j}(1-\gamma) - \underline{c}^{1-\gamma} \sum_{j=1}^n W_{t,j}(0)}{\underbrace{\sum_{j=1}^n q_j(t) c_{t,j}^{-\gamma} W_{t,j}(-\gamma)}_{\bar{v}(t)}}$$

Proof of Appendix Proposition B3: see end of appendix B

We also immediately obtain the following corollary:

Appendix Corollary B4:

$$\begin{aligned} VSI_{i,j} &= VSL_i - VSL_j \frac{q_j(0)c_{0,j}^{-\gamma}}{q_i(0)c_{0,i}^{-\gamma}} \\ &= VSL_i - \left(\frac{q_j(0)}{q_i(0)} \right) \left(\frac{c_{0,i}}{c_{0,j}} \right)^\gamma VSL_j \end{aligned}$$

Proofs for Appendix B3

Proof of Appendix Proposition B1:

The proof proceeds by induction on $t \leq T$. For the base case $t = T$, we set $K_{T,i} = q_i(T)$. For the induction step, suppose the proposition is true for case $t + 1$. We have

$$V(t, w, i) = \max_c \left\{ q_i(t) \frac{c^{1-\gamma}}{1-\gamma} + e^{-\rho} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) \frac{K_{t+1,j}}{1-\gamma} [(w - c)e^r]^{1-\gamma} \right\}$$

From the first-order condition we obtain:

$$q_i(t)c^{-\gamma} = e^{r-\rho} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) K_{t+1,j} [(w - c)e^r]^{-\gamma}$$

Rearranging yields

$$(q_i(t))^{-1/\gamma} c = (w - c)e^r e^{(\rho-r)/\gamma} (1 - \bar{d}_i(t))^{-1/\gamma} \left(\sum_{j=1}^n p_{ij}(t) K_{t+1,j} \right)^{-1/\gamma}$$

Rearranging further yields

$$\begin{aligned} c &= w \frac{e^{r+\frac{(\rho-r)}{\gamma}} (1 - \bar{d}_i(t))^{-1/\gamma} (\sum_{j=1}^n p_{ij}(t) K_{t+1,j})^{-1/\gamma}}{(q_i(t))^{-1/\gamma} + e^{r+\frac{(\rho-r)}{\gamma}} (1 - \bar{d}_i(t))^{-1/\gamma} (\sum_{j=1}^n p_{ij}(t) K_{t+1,j})^{-1/\gamma}} \\ &= w \frac{1}{1 + \underbrace{\left[\frac{e^{-\rho+r(1-\gamma)} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j})}{q_i(t)} \right]^{1/\gamma}}_{c_{t,i}}} \end{aligned}$$

Thus we obtain:

$$\begin{aligned} V(t, w, i) &= q_i(t) c_{t,i}^{1-\gamma} \frac{w^{1-\gamma}}{1-\gamma} + e^{-\rho} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) \frac{K_{t+1,j}}{1-\gamma} [w(1 - c_{t,i})e^r]^{1-\gamma} \\ &= \frac{w^{1-\gamma}}{1-\gamma} \left[q_i(t) c_{t,i}^{1-\gamma} + e^{(1-\gamma)r-\rho} (1 - c_{t,i})^{1-\gamma} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) K_{t+1,j} \right] \\ &= \frac{w^{1-\gamma}}{1-\gamma} \left[\frac{q_i(t) \left[e^{(1-\gamma)r-\rho} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) K_{t+1,j} \right]^{1-1/\gamma} + q_i(t) e^{(1-\gamma)r-\rho} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j})}{\left[(q_i(t))^{-1/\gamma} + \left[e^{(1-\gamma)r-\rho} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j}) \right]^{-1/\gamma} \right]^{1-\gamma}} \right] \end{aligned}$$

$$= \frac{w^{1-\gamma} q_i(t) e^{(1-\gamma)r-\rho} (1 - \bar{d}_i(t)) \sum_{j=1}^n p_{ij}(t) K_{t+1,j}}{1 - \gamma \left[(q_i(t))^{\frac{-1}{\gamma}} + \left[e^{(1-\gamma)r-\rho} (1 - \bar{d}_i(t)) (\sum_{j=1}^n p_{ij}(t) K_{t+1,j}) \right]^{\frac{-1}{\gamma}} \right]^{-\gamma}}$$

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Proof of Appendix Lemma B2:

$$\begin{aligned} W_{t+1,j}(\Psi) &= \mathbb{E} \left[\exp \left\{ - \int_0^{t+1} \mu(s) ds \right\} (w_{t+1})^\Psi \mathbf{1}\{Y_{t+1} = j\} \right] \\ &= \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} (w_t - c_t) e^r \mathbf{1}\{Y_{t+1} = j\} \exp \left\{ - \int_t^{t+1} \mu(s) ds \right\} \right] \\ &= \sum_{k=1}^n \mathbb{E} \left[\mathbf{1}\{Y_t = k\} \exp \left\{ - \int_0^t \mu(s) ds \right\} e^{r\Psi} w_t^\Psi (1 - c_{t,k})^\Psi \underbrace{\mathbb{E} \left[\mathbf{1}\{Y_{t+1} = j\} \exp \left\{ - \int_t^{t+1} \mu(s) ds \right\} \middle| Y_t = k \right]}_{(1 - \bar{d}_k(t)) p_{kj}(t)} \right] \\ &= e^{r\Psi} \sum_{k=1}^n W_{t,k}(Y) (1 - c_{t,k})^\Psi (1 - \bar{d}_k(t)) p_{kj}(t) \end{aligned}$$

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Proof of Appendix Proposition B3:

Note that we have

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_{Y_t}(t) c_{Y_t}^\Psi \right] &= \sum_{j=1}^n \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_{Y_t}(t) c_{Y_t}^\Psi \mathbf{1}\{Y_t = j\} \right] \\ &= \sum_{j=1}^n \mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} q_j(t) c_{t,j}^\Psi w_t^\Psi \mathbf{1}\{Y_t = j\} \right] \\ &= \sum_{j=1}^n q_j(t) c_{t,j}^\Psi \underbrace{\mathbb{E} \left[\exp \left\{ - \int_0^t \mu(s) ds \right\} w_t^\Psi \mathbf{1}\{Y_t = j\} \right]}_{W_{t,j}(\Psi)} \end{aligned}$$

The proof follows by setting $\Psi = 1 - \gamma$, 0, and $-\gamma$ in the expression for VSL.

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C. The fully annuitized value of life when mortality is stochastic

Even when mortality is stochastic, a complete annuities market allows the consumer to fully insure against mortality risk. We assume a full menu of actuarially fair annuities is available where consumers can choose consumption streams, $c_{Y_t}(t)$, that depend on the health state, Y_t . The consumer's maximization problem is:

$$\begin{aligned} & \max_{c_{Y_t}(t)} \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) u(c_{Y_t}(t), q_{Y_t}(t)) dt \middle| Y_0 \right] & (20) \\ \text{s. t. } & \mathbb{E} \left[\int_0^T e^{-rt} S(t) c_{Y_t}(t) dt \middle| Y_0 \right] = \mathbb{E} \left[W_0 + \int_0^T e^{-rt} S(t) m_{Y_t}(t) dt \middle| Y_0 \right] \equiv \bar{W}(0, Y_0) \end{aligned}$$

where the net present value of wealth and future earnings at time t in state i is $\bar{W}(t, i)$, and $S(t)$ is defined as before. Define the consumer's objective function at time u as:

$$J(u, i) = \mathbb{E} \left[\int_0^{T-u} e^{-\rho t} \exp \left\{ - \int_0^t \mu(u+s) ds \right\} u(c_{Y_{u+t}}(u+t), q_{Y_{u+t}}(u+t)) dt \middle| Y_u = i \right] \quad (21)$$

We can write the objective function (21) recursively as:

$$\begin{aligned} J(u, i) = & \int_0^{T-u} e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_i(u+s) + \sum_{j \neq i} \lambda_{ij}(u+s) ds \right\} \left(u(c_i(u+t), q_i(u+t)) \right. \\ & \left. + \sum_{j \neq i} \lambda_{ij}(u+t) J(u+t, j) \right) dt \end{aligned}$$

Similarly, current wealth at time u in state i , including the value of future labor income, pays for future consumption such that:

$$\begin{aligned} \bar{W}(u, i) = & \mathbb{E} \left[\int_0^{T-u} e^{-rt} \exp \left\{ - \int_0^t \mu(u+s) ds \right\} c_{Y_{u+t}}(u+t) dt \middle| Y_u = i \right] \\ = & \int_0^{T-u} e^{-rt} \exp \left\{ - \int_0^t \bar{\mu}_i(u+s) + \sum_{j \neq i} \lambda_{ij}(u+s) ds \right\} \left(c_i(u+t) + \sum_{j \neq i} \lambda_{ij}(u+t) \bar{W}(u+t, j) \right) dt \end{aligned}$$

This in turn implies

$$\frac{\partial \bar{W}(t, i)}{\partial t} = (r + \bar{\mu}_i(t)) \bar{W}(t, i) - c_i(t) + \sum_{j \neq i} \lambda_{ij}(t) [\bar{W}(t, i) - \bar{W}(t, j)]$$

Define the optimal value function as

$$V(t, \bar{W}_t, Y_t) = \max_{\{c_{Y_s}(s), s \geq t\}} \{J(t, Y_t)\}$$

where $\bar{W}_t = (\bar{W}(t, Y_1), \dots, \bar{W}(t, Y_n))$. Under conventional regularity conditions, we know that if V and its partial derivatives are continuous, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) system of equations:

$$\begin{aligned}
& (\rho + \bar{\mu}_i(t))V(t, \bar{W}_t, i) \\
&= \max_{c_i(t)} \left\{ u(c_i(t), q_i(t)) \right. \\
&+ \sum_{k=1}^n \frac{\partial V(t, \bar{W}_t, i)}{\partial \bar{W}(t, k)} \left[(r + \bar{\mu}_k(t)) \bar{W}(t, k) - c_k(t) + \sum_{l \neq k} \lambda_{kl}(t) [\bar{W}(t, k) - \bar{W}(t, l)] \right] \\
&+ \left. \frac{\partial V(t, \bar{W}_t, i)}{\partial t} + \sum_{j \neq i} \lambda_{ij}(t) [V(t, \bar{W}_t, j) - V(t, \bar{W}_t, i)] \right\}, 1 \leq i \leq n
\end{aligned} \tag{22}$$

We are interested in understanding how optimal consumption, and thus the value of life, changes over the life-cycle in this problem. Similarly to the uninsured case in the main text, we follow Parpas and Webster (2013), who demonstrate that it is possible to reformulate a stochastic optimization problem as a deterministic problem that takes $V(t, \bar{W}_t, j), j \neq i$, along with the corresponding optimal policies, as exogenous. This then allows us to apply the maximum principle and derive analytic expressions.

Appendix Lemma C1:

The optimal value function for $Y_0 = i, V(t, \bar{W}_t, i)$, for the following deterministic optimization problem also satisfies the HJB given by (22), for each $i \in \{1, \dots, n\}$:

$$\begin{aligned}
V_0(0, \bar{W}_0, i) &= \max_{c_i(t)} \left[\int_0^T e^{-\rho t} \bar{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) dt \right] \\
\text{s. t. } \frac{\partial \bar{W}(t, j)}{\partial t} &= (r + \bar{\mu}_j(t)) \bar{W}(t, j) - c_j(t) + \sum_{k \neq j} \lambda_{jk}(t) [\bar{W}(t, j) - \bar{W}(t, k)], j = 1, \dots, n
\end{aligned} \tag{23}$$

where $V(t, \bar{W}_t, j)$ and $c_j(t), j \neq i$, are taken as exogenous.

Proof of Appendix Lemma C1: see end of Appendix C

Following Bertsekas (2005), the Hamiltonian for the (deterministic) maximization problem (23) is:

$$\begin{aligned}
H(\bar{W}_t, c_i(t), p_t) &= e^{-\rho t} \bar{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) \\
&+ \sum_{k=1}^n p_t^{(k)} \left[(r + \bar{\mu}_k(t)) \bar{W}(t, k) - c_k(t) + \sum_{l \neq k} \lambda_{kl}(t) [\bar{W}(t, k) - \bar{W}(t, l)] \right]
\end{aligned} \tag{24}$$

where $p_t^{(k)}$ is the costate variable corresponding to wealth $\bar{W}(t, k)$.

Appendix Lemma C2:

The consumer's first-order condition for the Hamiltonian (24) for $Y_0 = i$ is

$$e^{(r-\rho)t} u_c(c_i(t), q_i(t)) = \theta \tag{25}$$

where $\theta = \partial V(0, \bar{W}_0, i) / \partial \bar{W}(0, i)$ is equal to the marginal utility of wealth.

Proof of Appendix Lemma C2: see end of Appendix C

To analyze the value of life, we again let $\delta(t)$ be a perturbation on the mortality rate with $\int_0^T \delta(t)dt = 1$. As in the deterministic case, we will first derive the marginal utility of the life extension associated with this perturbation.

Appendix Proposition C3:

The marginal utility of life extension takes the same form as in the deterministic case:

$$\left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} = \mathbb{E} \left[\int_0^T [e^{-\rho t} u(c_{Y_t}(t), q_{Y_t}(t)) + e^{-rt} \theta(m_{Y_t}(t) - c_{Y_t}(t))] \left(\int_0^t \delta(s) ds \right) S(t) dt \middle| Y_0 \right]$$

Proof of Appendix Proposition C3: see end of Appendix C

Choosing again the Dirac delta function for $\delta(\cdot)$ and dividing the result by the marginal utility of wealth, θ , yields the value of statistical life:

$$VSL = \mathbb{E} \left[\int_0^T e^{-rt} S(t) v_{Y_t}(t) dt \middle| Y_0 \right] \quad (26)$$

where the value of a statistical life-year is:

$$v_{Y_t}(t) = \frac{u(c_{Y_t}(t), q_{Y_t}(t))}{u_c(c_{Y_t}(t), q_{Y_t}(t))} + m_{Y_t}(t) - c_{Y_t}(t)$$

Comparing (26) to (3) reveals that stochastic mortality does not alter the basic expression for VSL . Consumers continue to discount future life-years by the rate of interest and by survival. One notable difference is that stochastic mortality generates variance in the value of life, which can now increase or decrease following the transition to a new health state.

We can obtain the life-cycle profile of consumption by differentiating the first-order condition (25) with respect to t . Doing so confirms that, as in the deterministic case, annuitization insulates consumption from mortality risk:¹⁸

$$\frac{\dot{c}_{Y_t}}{c_{Y_t}} = \frac{\dot{c}}{c} = \sigma(r - \rho) + \sigma\eta \frac{\dot{q}}{q}$$

Our results demonstrate that stochastic mortality, by itself, does not alter the basic insights regarding VSL offered by the prior literature as long as one maintains the assumption of full annuitization.

A novel feature of the stochastic model is that it permits an investigation into the value of prevention, i.e., the value of a reduction in the probability of transitioning to a different health state. This is not possible in a deterministic environment, where there is implicitly only one health state.

To analyze the value of prevention, let $\delta_{ij}(t)$ be a perturbation on $\lambda_{ij}(t)$, where $\sum_{j \neq i} \int_0^T \delta_{ij}(t)dt = 1$. As in the life-extension case, it is helpful to choose the Dirac delta function for $\delta(\cdot)$, so that the probability is

¹⁸ We assume—like all prior studies—that full indemnity healthcare insurance is available, which is equivalent to assuming that $q(t)$ is independent of the health state. Without this assumption, sudden decreases in q could cause the value of life to jump (Lakdawalla, Malani, and Reif 2017).

perturbed at $t = 0$ and remains unaffected otherwise. It is also helpful to consider a reduction in the transition probability for only one alternative state, j_0 , so that $\delta_{ij}(t) = 0 \forall j \neq j_0$.

Appendix Proposition C4:

Define the value of statistical illness, $VSI(i, j_0)$, to be the value of marginal reduction in the probability of transitioning to state j_0 when in state i . This value is equal to:

$$\begin{aligned}
 VSI(i, j_0) &= \mathbb{E} \left[\int_0^T e^{-rt} \left[\frac{u(c_{Y_t}(t), q_{Y_t}(t))}{u_c(c_{Y_t}(t), q_{Y_t}(t))} + m(t) - c(t) \right] S(t) dt \middle| Y_0 = i \right] \\
 &\quad - \mathbb{E} \left[\int_0^T e^{-rt} \left[\frac{u(c_{Y_t}(t), q_{Y_t}(t))}{u_c(c_{Y_t}(t), q_{Y_t}(t))} + m(t) - c(t) \right] S(t) dt \middle| Y_0 = j_0 \right] \\
 &= VSL(i) - VSL(j_0 | \bar{W}(0) = W^*)
 \end{aligned} \tag{27}$$

where W^* is the value of the annuity that was initially purchased in state i that promised the state-contingent consumption stream $c_{Y_t}^*(t)$:

$$W^* = \mathbb{E} \left[\int_0^T e^{-rt} S(t) c_{Y_t}^*(t) dt \middle| Y_0 = j_0 \right]$$

Proof of Appendix Proposition C4: see end of Appendix C

The notation in equation (27) indicates that VSL in state j_0 is evaluated under the assumption that the consumer's annuity was purchased when she was in state i . If life expectancy in state j_0 is lower than in state i , the value of the annuity to the consumer falls.

Proofs for Appendix C

Proof of Appendix Lemma C1:

Let $V(t, \bar{W}_t, j)$ and $c_j(t), j \neq i$, be taken as given (exogenous). Consider the deterministic optimization problem:

$$V(0, \bar{W}_0, i) = \max_{c_i(t)} \left\{ \int_0^T e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) dt \right\}$$

$$s. t. \frac{\partial \bar{W}(t, j)}{\partial t} = \left(r + \bar{\mu}_j(t) \right) \bar{W}(t, j) - c_j(t) + \sum_{k \neq j} \lambda_{jk}(t) [\bar{W}(t, j) - \bar{W}(t, k)], j = 1, \dots, n$$

Denote the optimal value-to-go as

$$\tilde{V}(u, \bar{W}_u, i) = \max_{c_i(t)} \left\{ \int_u^T e^{-\rho t} \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) dt \right\}$$

Setting $\tilde{V}(u, \bar{W}_u, i) = e^{-\rho t} \tilde{S}(i, t) V(t, \bar{W}_t, i)$ then demonstrates that $V(\cdot)$ satisfies the HJB (22) for i .

QED

Proof of Appendix Lemma C2:

The costate equations for the Hamiltonian (24) are:

$$\dot{p}_t^{(i)} = -p_t^{(i)} \left(r + \bar{\mu}_i(t) + \sum_{j \neq i} \lambda_{ij}(t) \right) + \sum_{l \neq i} \lambda_{li}(t) p_t^{(l)} \text{ and}$$

$$\dot{p}_t^{(k)} = e^{-\rho t} \tilde{S}(i, t) \lambda_{ik}(t) \frac{\partial V(t, \bar{W}_t, k)}{\partial \bar{W}(t, k)} - p_t^{(k)} \left(r + \bar{\mu}_k + \sum_{l \neq k} \lambda_{kl}(t) \right) + \sum_{l \neq k} \lambda_{lk}(t) p_t^{(l)}$$

for $k \neq i$. Suppose that $p_t^{(k)} = 0, k \neq i$. (We will verify this at the end of the proof.) This implies:

$$p_t^{(i)} = \theta e^{-rt} \tilde{S}(i, t)$$

where θ is a constant. Note also that the first-order condition of the Hamiltonian with respect to $c_i(t)$ is

$$e^{-\rho t} \tilde{S}(i, t) u_c(c_i(t), q_i(t)) = p_t^{(i)}$$

Setting these last two equations equal to each other then yields the desired result.

To verify that $p_t^{(k)} = 0, k \neq i$, note that the previous result implies via the HJB that $\partial V(t, \bar{W}_t, i) / \partial \bar{W}(t, i) = \theta e^{-(r-\rho)t}$, so that the costate equation for $k \neq i$ is

$$\begin{aligned} \dot{p}_t^{(k)} &= -\overbrace{\theta e^{-rt} \tilde{S}(i,t)}^{p_t^{(i)}} \lambda_{ik}(t) + p_t^{(i)} \lambda_{ik}(t) - p_t^{(k)} \left(r + \bar{\mu}_k + \sum_{l \neq k} \lambda_{kl}(t) \right) + \sum_{l \neq \{k,i\}} \lambda_{lk}(t) p_t^{(l)} \\ &= 0 \end{aligned}$$

QED

Proof of Appendix Proposition C3:

The marginal utility of life extension is defined as

$$\left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \mathbb{E} \left[\int_0^T e^{-\rho t} \exp \left\{ - \int_0^t \mu(s) - \varepsilon \delta(s) ds \right\} \left(u \left(c_{Y_t}^\varepsilon(t), q_{Y_t}(t) \right) \right) dt \middle| Y_0 \right] \Big|_{\varepsilon=0}$$

where $c^\varepsilon(t)$ represents the equilibrium variation in $c(t)$ caused by this perturbation. Then

$$\begin{aligned} \left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} &= \mathbb{E} \left[\int_0^T e^{-\rho t} \left(\int_0^t \delta(s) ds \right) \exp \left\{ - \int_0^t \mu(s) ds \right\} u \left(c_{Y_t}(t), q_{Y_t}(t) \right) dt \middle| Y_0 \right] \\ &\quad + \mathbb{E} \left[\int_0^T e^{-\rho t} \exp \left\{ - \int_0^t \mu(s) ds \right\} u_c \left(c_{Y_t}^\varepsilon(t), q_{Y_t}(t) \right) \frac{\partial c_{Y_t}^\varepsilon(t)}{\partial \varepsilon} \Big|_{\varepsilon=0} dt \middle| Y_0 \right] \\ &= \mathbb{E} \left[\int_0^T e^{-\rho t} \left(\int_0^t \delta(s) ds \right) \exp \left\{ - \int_0^t \mu(s) ds \right\} u \left(c_{Y_t}(t), q_{Y_t}(t) \right) dt \middle| Y_0 \right] \\ &\quad + \theta \mathbb{E} \left[\int_0^T e^{-rt} \exp \left\{ - \int_0^t \mu(s) ds \right\} \frac{\partial c_{Y_t}^\varepsilon(t)}{\partial \varepsilon} \Big|_{\varepsilon=0} dt \middle| Y_0 \right] \end{aligned}$$

Finally, the budget constraint implies

$$\begin{aligned} 0 &= \left. \frac{\partial W_0}{\partial \varepsilon} \right|_{\varepsilon=0} \\ &= \frac{\partial}{\partial \varepsilon} \mathbb{E} \left[\int_0^T e^{-rt} \exp \left\{ - \int_0^t \mu(s) - \varepsilon \delta(s) ds \right\} \left(c_{Y_t}^\varepsilon(t) - m_{Y_t}(t) \right) dt \middle| Y_0 \right] \Big|_{\varepsilon=0} \\ &= \mathbb{E} \left[\int_0^T e^{-rt} \left(\int_0^t \delta(s) ds \right) \exp \left\{ - \int_0^t \mu(s) ds \right\} \left(c_{Y_t}(t) - m_{Y_t}(t) \right) dt \middle| Y_0 \right] \\ &\quad + \mathbb{E} \left[\int_0^T e^{-rt} \exp \left\{ - \int_0^t \mu(s) ds \right\} \frac{\partial c_{Y_t}^\varepsilon(t)}{\partial \varepsilon} \Big|_{\varepsilon=0} dt \middle| Y_0 \right] \end{aligned}$$

Plugging this last result into the expression for $\left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0}$ then yields the desired result.

QED

Proof of Appendix Proposition C4:

Working from equation (23) in the text, the marginal utility of prevention is given by

$$\begin{aligned} \left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} \exp \left\{ - \int_0^t \bar{\mu}_i(s) + \sum_{j \neq i} [\lambda_{ij}(s) - \varepsilon \delta_{ij}(s)] ds \right\} \left(u(c_i^\varepsilon(t), q_i(t)) \right. \\ &\quad \left. + \sum_{j \neq i} [\lambda_{ij}(t) - \varepsilon \delta_{ij}(t)] V(t, \bar{W}_t^\varepsilon, j) \right) dt \Big|_{\varepsilon=0} \end{aligned}$$

where $c_i^\varepsilon(t)$ and \bar{W}_t^ε represent the equilibrium variations in $c_i(t)$ and \bar{W}_t caused by this perturbation. This yields

$$\begin{aligned} \left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} &= \int_0^T e^{-\rho t} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) \\ &\quad - e^{-\rho t} \tilde{S}(i, t) \sum_{j \neq i} \delta_{ij}(t) V(t, \bar{W}_t, j) \\ &\quad + e^{-\rho t} \tilde{S}(i, t) \left(\frac{u_c(c_i(t), q_i(t))}{\theta e^{-(r-\rho)t}} \frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} \Big|_{\varepsilon=0} + \sum_{j \neq i} \lambda_{ij}(t) \frac{V_W(t, \bar{W}_t, j)}{\theta e^{-(r-\rho)t}} \frac{\partial \bar{W}^\varepsilon(t, j)}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) dt \end{aligned}$$

Next, note that the budget constraint implies

$$\begin{aligned} 0 &= \left. \frac{\partial W_0^\varepsilon}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \int_0^T e^{-rt} \exp \left\{ - \int_0^t \bar{\mu}_i(s) + \sum_{j \neq i} [\lambda_{ij}(s) - \varepsilon \delta_{ij}(s)] ds \right\} \left(c_i^\varepsilon(t) - m_i(t) \right. \\ &\quad \left. + \sum_{j \neq i} [\lambda_{ij}(t) - \varepsilon \delta_{ij}(t)] \bar{W}^\varepsilon(t, j) \right) dt \Big|_{\varepsilon=0} \\ &= \int_0^T e^{-rt} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \tilde{S}(i, t) \left(c_i(t) - m_i(t) + \sum_{j \neq i} \lambda_{ij}(t) \bar{W}(t, j) \right) dt \\ &\quad - e^{-rt} \tilde{S}(i, t) \sum_{j \neq i} \delta_{ij}(t) \bar{W}(t, j) \\ &\quad + e^{-rt} \tilde{S}(i, t) \left(\frac{\partial c_i^\varepsilon(t)}{\partial \varepsilon} \Big|_{\varepsilon=0} + \sum_{j \neq i} \lambda_{ij}(t) \frac{\partial \bar{W}^\varepsilon(t, j)}{\partial \varepsilon} \Big|_{\varepsilon=0} \right) dt \end{aligned}$$

Substituting in then yields the final result for the marginal utility of the reduction in this transition intensity:

$$\begin{aligned}
\left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} &= \int_0^T e^{-\rho t} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \tilde{S}(i, t) \left(u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) \\
&\quad - e^{-\rho t} \tilde{S}(i, t) \sum_{j \neq i} \delta_{ij}(t) V(t, \bar{W}_t, j) \\
&\quad - \theta e^{-rt} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \tilde{S}(i, t) \left(c_i(t) - m_i(t) + \sum_{j \neq i} \lambda_{ij}(t) \bar{W}(t, j) \right) \\
&\quad + \theta e^{-rt} \tilde{S}(i, t) \sum_{j \neq i} \delta_{ij}(t) \bar{W}(t, j) dt \\
&= \int_0^T \left(e^{-\rho t} u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right. \\
&\quad \left. + \theta e^{-rt} \left(m_i(t) - c_i(t) - \sum_{j \neq i} \lambda_{ij}(t) \bar{W}(t, j) \right) \right) \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s) ds \right) \tilde{S}(i, t) \\
&\quad - \left(e^{-\rho t} \sum_{j \neq i} \delta_{ij}(t) V(t, \bar{W}_t, j) - \theta e^{-rt} \sum_{j \neq i} \delta_{ij}(t) \bar{W}(t, j) \right) \tilde{S}(i, t) dt
\end{aligned}$$

The first term inside the integral of the above expression represents the gain in marginal utility from a reduction in the probability of exiting state $Y_t = i$, and is analogous to the expression for $\left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0}$ for life-extension. The second term represents the loss in marginal utility from the reduction in probability of transitioning to other possible states. If these other states correspond to lower health (utility) than state i , then the net effect on marginal utility is positive.

Next, we choose the Dirac delta function for $\delta(\cdot)$, so that the probability is perturbed at $t = 0$ and remains unaffected otherwise. We also consider a reduction in the transition probability for only one alternative state, j_0 , so that $\delta_{ij}(t) = 0 \forall j \neq j_0$. This simplifies the above expression to

$$\begin{aligned}
\left. \frac{\partial V / \partial \varepsilon}{\partial V / \partial W} \right|_{\varepsilon=0} &= \int_0^T \left(e^{-\rho t} u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) + \theta e^{-rt} \left(m_i(t) - c_i(t) - \sum_{j \neq i} \lambda_{ij}(t) \bar{W}(t, j) \right) \right) \tilde{S}(i, t) \\
&\quad - \left(e^{-\rho t} V(t, \bar{W}_t, j_0) - \theta e^{-rt} \bar{W}(t, j_0) \right) \tilde{S}(i, t) dt \\
&= \int_0^T \left(e^{-\rho t} \tilde{S}(i, t) u(c_i(t), q_i(t)) + \sum_{j \neq i} \lambda_{ij}(t) V(t, \bar{W}_t, j) \right) dt - V(0, \bar{W}_t, j_0) \\
&\quad + \theta \left[\int_0^T \left(e^{-rt} \left(m_i(t) - c_i(t) - \sum_{j \neq i} \lambda_{ij}(t) \bar{W}(t, j) \right) \right) \tilde{S}(i, t) dt + \bar{W}(0, j_0) \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) u(c_{Y_t}(t), q_{Y_t}(t)) dt \middle| Y_0 = i \right] - \mathbb{E} \left[\int_0^T e^{-\rho t} S(t) u(c_{Y_t}(t), q_{Y_t}(t)) dt \middle| Y_0 = j_0, W_0^* = W^{new} \right] \\
&\quad + \theta \left(\mathbb{E} \left[\int_0^T e^{-rt} S(t) (m_{Y_t}(t) - c_{Y_t}(t)) dt \middle| Y_0 = i \right] \right. \\
&\quad \left. - \mathbb{E} \left[\int_0^T e^{-rt} S(t) (m_{Y_t}(t) - c_{Y_t}(t)) dt \middle| Y_0 = j_0, W_0^* = W^{new} \right] \right)
\end{aligned}$$

where W^{new} represents the change in value of the annuity menu purchased in state i when immediately jumping to state j_0 . Dividing the above expression by the marginal utility of wealth, given by (25), then yields (27), the value of statistical illness (VSI).

QED